

METASTABILITY FOR PARABOLIC EQUATIONS WITH DRIFT: PART II. THE QUASILINEAR CASE

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ABSTRACT. This is the second part of our series of papers on metastability results for parabolic equations with drift. The aim is to present a self contained study, using partial differential equations methods, of the metastability properties of quasi-linear parabolic equations with a drift and to obtain results similar to those in Freidlin and Koralov [6, 8].

NOTATION. We work in \mathbb{R}^n and write \mathbb{S}^n for the space of real $n \times n$ symmetric matrices. For any $\theta \in (0, 1]$, $\mathbb{S}^n(\theta)$ denotes the subset of all $a \in \mathbb{S}^n$ satisfying $\theta I \leq a \leq \theta^{-1}I$, where I is the $n \times n$ identity matrix. If $a \in \mathbb{S}^n$, then $\text{tr } a$ denotes its trace, and, for $a, b \in \mathbb{S}^n$, $a \leq b$ if and only if $b - a$ is a nonnegative matrix. Given $p \in \mathbb{R}^n$, let $p \otimes p$ denote the symmetric matrix $(p_i p_j)_{1 \leq i, j \leq n}$. If U is a subset of \mathbb{R}^k for some $k \in \mathbb{N}$, then $C(U; \mathbb{S}^n(\theta))$ is the set of $\mathbb{S}^n(\theta)$ -valued continuous maps from U into \mathbb{S}^n . For $a \in \mathbb{S}^n$ and $p \in \mathbb{R}^n$, $ap \cdot p := \sum_{i,j=1}^n a_{ij} p_j p_i$. If $r_1, r_2 \in \mathbb{R}$, then $r_1 \wedge r_2 := \min\{r_1, r_2\}$ and $r_1 \vee r_2 := \max\{r_1, r_2\}$ and, for $r \in \mathbb{R}$, $r_+ = r \vee 0$ and $r_- = (-r) \vee 0$. We use the convention $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. The open ball in \mathbb{R}^n with radius $R > 0$ and center at $x \in \mathbb{R}^n$ is $B_R(x)$, and $B_R := B_R(0)$. Given $\Omega \subset \mathbb{R}^n$ and $\delta > 0$, we write $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}$, and, for $T > 0$, $Q_T := \Omega \times (0, T)$; if $T = \infty$, then we write Q instead of Q_∞ . The parabolic boundary of Q_T is $\partial_p Q_T := (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$. We denote by $\text{Lip}(A, \mathbb{R}^k)$ the set of the \mathbb{R}^k -valued Lipschitz continuous functions defined in $A \subset \mathbb{R}^k$; when $k = 1$, we often write $\text{Lip}(A)$. We write $\text{USC}(A)$ and $\text{LSC}(A)$ for the set of, respectively, upper and lower semicontinuous functions defined on A , and, when $A \subset \mathbb{R}^n \times [0, \infty)$ is open, $C^{2,1}(A)$ is the space of functions which are continuously differentiable twice with respect to the space variables and once with respect to the time variable. Given a bounded family of functions $f_\delta : A \rightarrow \mathbb{R}$, $\limsup_{\delta \rightarrow 0}^* f_\delta(x) := \lim_{r \rightarrow 0} \sup\{f_\delta(x+y) : x+y \in A, |y| + \delta \leq r\}$ and $\liminf_{\delta \rightarrow 0}^* f_\delta(x) := \lim_{r \rightarrow 0} \inf\{f_\delta(x+y) : x+y \in A, |y| + \delta \leq r\}$. If A is a closed subset of \mathbb{R}^n and $f : A \rightarrow \mathbb{R}$, $\arg \min(f|A) := \{x \in A : f(x) = \min_{y \in A} f(y)\}$. We use C to denote constants, which may change from line to line. When we want to display the dependence of a constant C on a parameter a , we write $C = C(a)$, and, for $a, b \in \mathbb{R}$, $a \approx b$ means that a and b are close to each other in a controlled way. Finally to simplify the notation we write $\{a_k\}$ to denote the sequence $\{a_k\}_{k \in \mathbb{N}}$.

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1. INTRODUCTION

This is the second part of our series of papers on metastability results for parabolic equations with drift. The aim is to present a self contained study, using partial differential equations (pde for short) methods, of the metastability properties of quasi-linear parabolic equations with a drift and to obtain results similar to those in Freidlin and Koralov [6, 8].

More precisely we are interested in the asymptotic behavior, as $\varepsilon \rightarrow 0$ and $t \rightarrow \infty$, of the solution $u^\varepsilon = u^\varepsilon(x, t)$ of the initial-boundary value problem

$$(1.1) \quad u_t^\varepsilon = \varepsilon \operatorname{tr}[a(x, u^\varepsilon) D^2 u^\varepsilon] + b(x) \cdot Du^\varepsilon \quad \text{in } Q,$$

and

$$(1.2) \quad u^\varepsilon = g \quad \text{on } \partial_p Q,$$

where

$$(1.3) \quad \Omega \text{ is a bounded } C^1\text{-domain with outward normal vector } \nu$$

and

$$(1.4) \quad g \in C(\overline{\Omega}).$$

Throughout the paper we assume that, for some $\theta_0 \in (0, 1]$,

$$(1.5) \quad a \in C(\overline{\Omega} \times \mathbb{R}; \mathbb{S}^n(\theta_0)),$$

and

$$(1.6) \quad b \in \operatorname{Lip}(\mathbb{R}^n; \mathbb{R}^n) \quad \text{with } b(0) = 0$$

is such that

$$(1.7) \quad \begin{array}{l} \text{the origin is a (unique) globally asymptotically stable point of} \\ \text{the dynamical system } \dot{X} = b(X) \text{ generated by } b. \end{array}$$

This last assumption is further quantified by the additional requirements that b points inward at the boundary points of Ω , that is,

$$(1.8) \quad b \cdot \nu < 0 \quad \text{on } \partial\Omega,$$

and there exist $b_0 > 0$ and $r_0 > 0$ such that $\overline{B}_{r_0} \subset \Omega$, and

$$(1.9) \quad b(x) \cdot x \leq -b_0|x|^2 \quad \text{for all } x \in B_{r_0}.$$

For later use we summarize all the above assumptions in the list

$$(1.10) \quad (1.3), (1.4), (1.5), (1.6), (1.7), (1.8) \text{ and } (1.9).$$

The asymptotic behavior of the u^ε 's is described in Theorem 1. Our arguments are based entirely on pde methods and the main tools are the comparison principle and the construction of two kinds of barrier functions for parabolic equations. The later was the main subject of our previous paper [11].

We work with either classical or viscosity solutions depending on the context and most of the times we say solution without making a distinction. When we write inequalities for viscosity sub- or super-solutions, we use the \leq and \geq signs for a sub- and super-solutions respectively. Finally, we will always work with $\varepsilon \in (0, 1)$ and we will not repeat this.

An important tool is the quasi-potential V^c associated, for each $c \in \mathbb{R}$, with $(a(\cdot, c), b)$, which is characterized by the property

$$V^c \text{ is the maximal subsolution of } H^c(x, Du) = 0 \text{ in } \Omega \text{ and } u(0) = 0,$$

where the Hamiltonian $H^c \in C(\overline{\Omega} \times \mathbb{R}^n)$ is given by

$$H^c(x, p) := a(x, c)p \cdot p + b(x) \cdot p.$$

The quasi-potential V^c is also the unique (viscosity) solution $u \in \text{Lip}(\overline{\Omega})$ of the state-constraints problem for the Hamilton-Jacobi equation $H(x, Du) = 0$ in Ω , with the additional condition that $u(0) = 0$. (See Lemma C.1 in Appendix C for the uniqueness of this state-constraints problem, and also Soner [14], Fleming and Soner [5] and Ishii [10] for some related aspects.)

Next we introduce some terminology and introduce some additional notation and hypotheses similar to those in [6, 8].

Consider the map $M : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(1.11) \quad M(c) := \min_{\partial\Omega} V^c.$$

The continuity of a and the stability properties of viscosity solutions yield that the functions $\mathbb{R} \ni c \mapsto M(c)$ and $\overline{\Omega} \times \mathbb{R} \ni (x, c) \mapsto V^c(x) \in \mathbb{R}$ are continuous. The continuity of the latter is an easy consequence of the uniqueness of the state constraints problem.

Given $g \in C(\overline{Q})$, we set

$$c_0 := g(0), \quad g_{\min} := \min_{\overline{\Omega}} g, \quad g_{\max} := \max_{\overline{\Omega}} g, \quad g_1 := \min_{\partial\Omega} g, \quad g_2 := \max_{\partial\Omega} g,$$

and note that $[g_1, g_2] \subset [g_{\min}, g_{\max}]$. Henceforth we write

$$I_g := [g_{\min}, g_{\max}],$$

and we introduce the multi-valued map $G : I_g \rightarrow 2^{\mathbb{R}}$ by

$$G(c) := \{g(x) : x \in \arg \min(V^c|_{\partial\Omega})\}.$$

It is immediate that $G(c) \subset I_g$ for all $c \in I_g$. Moreover, since $(c, x) \mapsto V^c(x)$ and g are continuous on $\mathbb{R} \times \partial\Omega$ and $\partial\Omega$ respectively, it is easily checked that G is upper semicontinuous on I_g and, hence, $G(c)$ is compact for all $c \in I_g$.

Next we define the functions $G^{\pm} : I_g \rightarrow I_g$ by

$$G^+(c) := \max G(c) \quad \text{and} \quad G^-(c) := \min G(c).$$

and note that they are respectively upper and lower semicontinuous, and, moreover,

$$G^+(c) = \max_{\arg \min(V^c|_{\partial\Omega})} g \quad \text{and} \quad G^-(c) = \min_{\arg \min(V^c|_{\partial\Omega})} g.$$

Following [6, 8], we assume that

$$(1.12) \quad G^+(c_0) = G^-(c_0),$$

and set

$$g_0 := G^+(c_0) = G^-(c_0).$$

This assumption means that the set $G(c_0)$ is a singleton, that is,

$$g(x) = g_0 \quad \text{for all } x \in \arg \min(V^{c_0}|_{\partial\Omega}).$$

Next we define c_1 as follows:

$$(1.13) \quad \begin{cases} \text{if } g_0 \geq c_0, \text{ then } c_1 := \inf\{c \in [c_0, \infty) : G^-(c) \leq c\}, \text{ and,} \\ \text{if } g_0 \leq c_0, \text{ then } c_1 := \sup\{c \in (-\infty, c_0] : G^+(c) \geq c\}. \end{cases}$$

Note that, since $G(I_g) := \bigcup_{c \in I_g} G(c) \subset [g_1, g_2]$, we always have $c_1 \in [g_1, g_2]$ and observe that

$$(1.14) \quad \begin{cases} \text{if } c_1 > c_0, \text{ then } G^-(c) > c \text{ for all } c \in [c_0, c_1), \\ \text{if } c_1 < c_0, \text{ then } G^+(c) < c \text{ for all } c \in (c_1, c_0]. \end{cases}$$

We assume that the graph of G crosses the diagonal from the left to the right at c_1 , that is

$$(1.15) \quad \begin{cases} \text{for all } \delta_0 > 0, \text{ there exists } \delta \in (0, \delta_0] \text{ such that} \\ \quad \text{if } c_0 \geq c_1 > g_{\min}, \text{ then } G^-(c_1 - \delta) > c_1 - \delta, \\ \quad \text{if } c_0 \leq c_1 < g_{\max}, \text{ then } G^+(c_1 + \delta) < c_1 + \delta, \end{cases}$$

and we define the function $\bar{c} : (0, \infty) \rightarrow I_g$ as follows: For each $\lambda \in (0, \infty)$,

$$(1.16) \quad \bar{c}(\lambda) := \begin{cases} c_0 & \text{if either } \lambda < M(c_0) \text{ or } c_1 = c_0, \\ \min(c_1, \inf\{c \in [c_0, c_1] : M(c) = \lambda\}) & \text{if } \lambda \geq M(c_0) \text{ and } c_1 > c_0, \\ \max(c_1, \sup\{c \in [c_1, c_0] : M(c) = \lambda\}) & \text{if } \lambda \geq M(c_0) \text{ and } c_1 < c_0. \end{cases}$$

For later use we summarize the above assumptions in the list

$$(1.17) \quad (1.12) \text{ and } (1.15).$$

Since the definition of $\bar{c}(\lambda)$ is cumbersome, for clarity and to compare with the linear problem, we discuss what happens when $a(x, c)$ is independent of c . In this case the quasi-potential V and, hence, its minimum value $M = \min_{\partial\Omega} V$ do not depend on c , and the multi-valued map G is a constant. Assumption (1.12) then states that $g_0 = \min_{\arg \min(V|\partial\Omega)} g = \max_{\arg \min(V|\partial\Omega)} g$ and $G(c) = \{g_0\}$ and $G^-(c) = G^+(c) = g_0$ for all $c \in I_g$. It is easily checked that, if $g(0) = g_0$, then $\bar{c}(\lambda) = g(0) = g_0$ for all $\lambda > 0$, and, if either $g(0) < g_0$ or $g(0) > g_0$,

$$\bar{c}(\lambda) = \begin{cases} c_0 & \text{if } \lambda \leq M, \\ c_1 & \text{if } \lambda > M, \end{cases}$$

while, if $g(0) \neq g_0$, then $\bar{c}(\lambda)$ is discontinuous at $\lambda = M$.

The main result, which is similar to [6, Theorem 3.1; 8], is:

Theorem 1. *Assume (1.10) and (1.17) and let $\lambda > 0$ be a point of continuity of \bar{c} . If, for $\varepsilon \in (0, 1)$, $u^\varepsilon \in C(\bar{Q}) \cap C^{2,1}(Q)$ is a solution of (1.1) and (1.2), then, for each $\delta > 0$ so that $\Omega_\delta \neq \emptyset$,*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(\cdot, \exp(\lambda/\varepsilon)) = \bar{c}(\lambda) \text{ uniformly in } \Omega_\delta.$$

In view of the previous discussion, when $a(x, c)$ is independent of c , that is for linear equations, Theorem 1 is the same as [11, Theorem 1], except if $g(0) = g_0$. In this case, [11, Theorem 1] asserts, in addition, the uniform convergence of $u^\varepsilon(\cdot, \exp(\lambda/\varepsilon))$ on any compact subset of $\Omega \cup \arg \min(V|\partial\Omega)$.

As in [6, 8], to prove Theorem 1 we need to show the following three propositions, which were proved in [8] using large deviation results from [9]. The first two together state [8, Lemma 3.11], while the third is an observation which is very crucial for the proof of Lemma 6 (see [8, Lemma 3.12]).

Proposition 2. Assume (1.10) and let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (1.1). Assume furthermore that the u^ε 's are bounded on Q uniformly on ε and suppose that there exist sequences $\{\mu_k\}, \{\lambda_k\} \subset (0, \infty)$ and $\{\varepsilon_k\} \subset (0, 1)$ and constants $0 < a_1 < a_2$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, and, for all $k \in \mathbb{N}$,

$$0 < a_1 \leq \mu_k < \lambda_k \leq a_2, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

If $\beta_1 \neq \beta_2$, then $\limsup_{k \rightarrow \infty} \lambda_k \geq M(\beta_2)$.

Proposition 3. Assume (1.10) and let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (1.1) and (1.2). Assume further that there exist sequences $\{\mu_k\}, \{\lambda_k\} \subset (0, \infty)$ and $\{\varepsilon_k\} \subset (0, 1)$ and constants $0 < a_1 < a_2$ and $\beta_1, \beta_2 \in I_g$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, and, for all $k \in \mathbb{N}$,

$$0 < a_1 \leq \mu_k < \lambda_k \leq a_2, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

If $\beta_1 < \beta_2$, then $G^+(\beta_2) \geq \beta_2$, and, if $\beta_2 < \beta_1$, then $G^-(\beta_2) \leq \beta_2$.

Proposition 4. Assume (1.10) and let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (1.1) and (1.2). Fix $\beta_0 \in I_g$ and $\rho_0 > 0$, and assume that, for any $\delta > 0$, there exist $\gamma > 0$ and a sequence $\{\varepsilon_k\} \subset (0, 1)$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and, for all $\rho \in [\rho_0 - \gamma, \rho_0 + \gamma]$ and $k \in \mathbb{N}$,

$$(1.18) \quad u^{\varepsilon_k}(0, \exp(\rho/\varepsilon_k)) \in [\beta_0 - \delta, \beta_0 + \delta].$$

If either

$$(1.19) \quad G^-(\beta_0) > \beta_0 \quad \text{or} \quad G^+(\beta_0) < \beta_0,$$

then $\rho_0 \leq M(\beta_0)$.

We discuss next some of the new ideas that are needed to prove the main theorem.

Recall that we are interested in the asymptotic behavior, as $(\varepsilon, t) \rightarrow (0, \infty)$, of the solution u^ε of (1.1) and (1.2) in a logarithmic time scale, that is, in the behavior, as $\varepsilon \rightarrow 0$, of $u^\varepsilon(x, \exp(\lambda/\varepsilon))$ for any fixed $\lambda > 0$. It turns out that this is a consequence of what we call “uniform asymptotic constancy” which yields that, as $t \rightarrow \infty$, $u^\varepsilon(\cdot, t)$ behaves similarly to $u^\varepsilon(0, t)$ in the space $C(\Omega)$ equipped with the locally uniform convergence topology,

The uniform asymptotic constancy (see Theorem 10 below) is a crucial observation that goes beyond [11]. Roughly it says that, if u^ε is a bounded solution of (2.1), then, as $\varepsilon \rightarrow 0$, for any compact $K \subset \Omega$ and $\delta > 0$,

$$u^\varepsilon(x, t) \approx u^\varepsilon(0, t) \quad \text{uniformly for } (x, t) \in K \times [e^{\delta/\varepsilon}, \infty).$$

With this fact at hand the main theorem (Theorem 1) is an easy consequence of Propositions 2, 3 and 4.

Their proofs are based on the comparison (or maximum) principle and, thus, on the construction of barriers, that is sub- and super-solutions of (1.1). We have already built such functions in our previous work [11], where the matrix $a(x, c)$ is independent of c . Here (see Proposition 13 and Corollary 14) we modify the construction of one class of barrier functions in order to make the comparison argument straightforward.

The building block of the barrier functions in [11] and here is viscosity solutions of $H_\alpha(x, Du) = 0$ with some additional normalization conditions, where $\alpha \in C(\overline{\Omega}; \mathbb{S}^n(\theta_0))$ is selected as explained below and $H_\alpha(x, p) := \alpha(x)p \cdot p + \underline{\alpha}(x) \cdot p$. An important observation is that, if V_α is the quasi-potential associated with (α, b) , then $V_\alpha > 0$ in $\overline{\Omega} \setminus \{0\}$ and $M_\alpha := \min_{\partial\Omega} V_\alpha > 0$.

The barriers $w^\varepsilon : \bar{Q} \rightarrow \mathbb{R}$ are supersolutions of (1.1) of the form

$$w^\varepsilon(x, t) := \exp\left(\frac{v(x) - m}{\varepsilon}\right) + d_\varepsilon t,$$

where m and d_ε are positive constants such that $0 < m < M_\alpha$ and $d_\varepsilon = \exp(-\lambda_\varepsilon/\varepsilon)$ for some $\lambda_\varepsilon \approx m$, and v is an appropriately chosen smooth approximation of V_α . The choice of m yields that, for ε sufficiently small, w^ε is compatible with the Dirichlet data g on $\partial\Omega \times [0, \infty)$.

In view of the fact that a priori we have little knowledge of the uniform in ε regularity of solutions of (1.1), given such a solution u^ε , we treat $a^\varepsilon = a(x, u^\varepsilon(x, t))$ as an arbitrary element of $C(\bar{Q}; \mathbb{S}^n(\theta_0))$.

To motivate the choice of α in the construction of the barrier function given the a^ε above we compute in Q

$$\begin{aligned} w_t^\varepsilon - \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 w^\varepsilon] - b \cdot Dw^\varepsilon \\ = d_\varepsilon - \varepsilon^{-1} \exp\left(\frac{v(x) - m}{\varepsilon}\right) (H_\varepsilon(x, t, Dv) + \varepsilon \operatorname{tr}[a^\varepsilon D^2 v]) \end{aligned}$$

with $H_\varepsilon(x, t, p) := a^\varepsilon(x, t)p \cdot p + b(x) \cdot p$.

If $\alpha \in C(\bar{\Omega}; \mathbb{S}^n(\theta_0))$ satisfies $a^\varepsilon \leq \alpha$ in Q , then

$$w_t^\varepsilon - \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 w^\varepsilon] - b \cdot Dw^\varepsilon \geq d_\varepsilon - \varepsilon^{-1} \exp\left(\frac{v(x) - m}{\varepsilon}\right) (H_\alpha(x, Dv) + O(\varepsilon)) \geq 0,$$

with the last the inequality holding, if ε is sufficiently small, because of the choice of v and d_ε —the details are given in Proposition 12.

A very important fact in our analysis (see Proposition 11 below for the precise statement) is that the locally uniform convergence topology of $C(\Omega)$ is strong enough to imply that, if $\alpha(x) \approx a(x, c)$ in $C(\Omega)$, then $M_\alpha \approx M(c)$ and $\arg \min(V_\alpha | \partial\Omega) \approx \arg \min(V^c | \partial\Omega)$.

To describe the idea which is in the core of the proof of, for example, Proposition 2, we consider the very special case that, for $\varepsilon > 0$ sufficiently small and some constants $c, \gamma > 0$ and $0 < \delta < \mu < \lambda$,

$$|u^\varepsilon(0, t) - c| < \gamma \quad \text{for all } t \in [\exp(\delta/\varepsilon), \exp(\lambda/\varepsilon)],$$

and

$$u^\varepsilon(0, \exp(\delta/\varepsilon)) = c \quad \text{and} \quad u^\varepsilon(0, \exp(\mu/\varepsilon)) > c + \eta \quad \text{for some } \eta \in (0, \gamma).$$

We then choose $\alpha \in C(\bar{\Omega}; \mathbb{S}^n(\theta_0))$ so that $a^\varepsilon \leq \alpha$ in $\Omega \times [t_\varepsilon, T_\varepsilon]$, where $t_\varepsilon := \exp(\delta/\varepsilon)$ and $T_\varepsilon := \exp(\lambda/\varepsilon)$. Using the barrier w^ε as in the linear case (see [11, Theorem 1 (i)]), we conclude that, as $\varepsilon \rightarrow 0$, for any $\rho < M_\alpha$, $u^\varepsilon(0, t) \rightarrow c$ for all $t \in [t_\varepsilon, T_\varepsilon \wedge \exp(\rho/\varepsilon)]$, which implies that $\mu \geq M_\alpha$. Furthermore, according to the previous arguments, α can be chosen, so that, as $\gamma \rightarrow 0$, $M_\alpha \rightarrow M^c$.

Organization of the paper. The rest of the paper is organized as follows. In Section 2 we study the asymptotic constancy, that is the effect of the drift term in parabolic equations like (1.1). In Section 3 we introduce Hamilton-Jacobi equations related to (1.1), which have quadratic nonlinearity, and study the continuity properties of the associated quasi-potentials. Section 4 is devoted to the construction of two kind of barrier functions, or sub- and super-solutions, which are used to study the asymptotic behavior of solutions of linear parabolic equations, that is equations like (1.1) with $a \in C(\bar{Q}; \mathbb{S}^n(\theta_0))$. The proofs

of Propositions 2, 3 and 4 and Theorem 1 are given in Sections 5 and 6 respectively. Some basic properties of viscosity solutions are explained in Appendices A, B and C.

2. THE ASYMPTOTIC CONSTANCY

We consider the linear pde

$$(2.1) \quad u_t^\varepsilon = \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 u^\varepsilon] + b(x) \cdot Du^\varepsilon \quad \text{in } Q.$$

We assume, in addition to (1.6) and (1.9), that

$$(2.2) \quad a^\varepsilon \in C(\overline{Q}, \mathbb{S}^n(\theta_0)).$$

The goal here is to show that the drift term in (2.1) has a strong effect to propagate, as $\varepsilon \rightarrow 0$, the values of the solutions u^ε at $x = 0$ to Ω ; for future reference we call this fact the asymptotic constancy.

It turns out that the asymptotic constancy does not depend on any properties of a^ε other than (2.2). It is, therefore, technically more convenient to study, in some instances, instead of (2.1), the problem

$$(2.3) \quad v_t = \varepsilon P^+(D^2 v) + b(x) \cdot Dv \quad \text{in } Q,$$

where P^+ is the Pucci operator associated with $\mathbb{S}^n(\theta_0)$ defined by

$$(2.4) \quad P^+(X) = \sup\{\operatorname{tr}[AX] : A \in \mathbb{S}^n(\theta_0)\},$$

which is, obviously, uniformly elliptic with constants θ_0 and θ_0^{-1} , that is, for all matrices $X, Y \in \mathbb{S}^n$ such that $X \leq Y$,

$$(2.5) \quad \theta_0 \operatorname{tr}(Y - X) \leq P^+(Y) - P^+(X) \leq \theta_0^{-1} \operatorname{tr}(Y - X).$$

Some useful barrier functions. We fix an auxiliary function $h \in C^2([0, \infty))$ with the properties

$$(2.6) \quad 0 \leq h \leq 1, \quad h = 0 \quad \text{in } [0, 1/2], \quad h = 1 \quad \text{in } [1, \infty) \quad \text{and } h' \geq 0,$$

set

$$k := b_0/2 \quad \text{and} \quad R_0 := 2\sqrt{2n}/\sqrt{b_0\theta_0},$$

choose $R \in [R_0, \infty)$, $r \in (0, r_0]$, where r_0 is as in (1.9), and $\varepsilon_0 \in (0, 1)$ so that

$$(2.7) \quad \sqrt{\varepsilon_0} R < r,$$

and, for $\varepsilon \in (0, \varepsilon_0]$, let

$$(2.8) \quad \tau = \tau(\varepsilon) := \frac{1}{k} \log \left(\frac{r}{R\sqrt{\varepsilon}} \right).$$

With all these choices at hand we introduce the functions $p^\varepsilon, q^\varepsilon : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ defined by

$$(2.9) \quad p^\varepsilon(x, t) := h((R\sqrt{\varepsilon})^{-1}|x| e^{-kt})$$

and

$$(2.10) \quad q^\varepsilon(x, t) := p^\varepsilon(x, t) + \frac{\|h''\|_{L^\infty}}{R^2\theta_0} \int_0^t e^{-2ks} ds;$$

observe that, since h vanishes identically in a neighborhood of the positive time axis $l := \{0\} \times (0, \infty)$, p^ε and q^ε are smooth in $\mathbb{R}^n \times (0, \infty)$.

We note that p^ε appears in the proof of [6, Lemma 3.6; 8]. The difference is that these references consider equations like (2.1), while here we study (2.3).

The following lemma summarizes the properties of q^ε . Its proof is based on long explicit but also straightforward calculations. The reader may want to skip the details on first reading.

Lemma 1. *Assume (1.6), (1.9) and (2.5). With the above choices of k , R , r , ε_0 , ε and τ , the function q^ε given by (2.10) is a supersolution to (2.3) in $B_{r_0} \times (0, \infty)$. Moreover,*

$$\begin{cases} q^\varepsilon(\cdot, 0) \geq 0 & \text{in } B_r, \quad q^\varepsilon(\cdot, 0) \geq 1 & \text{in } B_r \setminus B_{\sqrt{\varepsilon}R}, \\ q^\varepsilon \geq 1 & \text{in } \partial B_r \times [0, \tau] \quad \text{and} \quad q^\varepsilon(\cdot, \tau) \leq \frac{\|h''\|_{L^\infty}}{b_0\theta_0 R^2} & \text{on } B_{r/2}. \end{cases}$$

Proof. First note that

$$p^\varepsilon(x, t) = 1 \quad \text{if } |x| \geq R\sqrt{\varepsilon}e^{kt} \quad \text{and} \quad p^\varepsilon(x, t) = 0 \quad \text{if } |x| \leq \frac{1}{2}R\sqrt{\varepsilon}e^{kt}.$$

For $(x, t) \in B_{r_0} \times (0, \infty)$ we write $\rho = \frac{1}{R\sqrt{\varepsilon}}$, $r_{x,t} = (R\sqrt{\varepsilon})^{-1}|x|e^{-kt}$ and $\bar{x} := x/|x|$ (since, in view of the above, p^ε vanishes in a neighborhood of the origin we do not have to be concerned about $x = 0$), and find

$$\begin{aligned} p_t^\varepsilon(x, t) &= -kh'(r_{x,t})|x|\rho e^{-kt}, & Dp^\varepsilon(x, t) &= h'(r_{x,t})\rho\bar{x}e^{-kt}, \\ D^2p^\varepsilon(x, t) &= h'(r_{x,t})\rho e^{-kt} \frac{1}{|x|}(I - \bar{x} \otimes \bar{x}) + h''(r_{x,t})\rho^2 e^{-2kt} \bar{x} \otimes \bar{x}. \end{aligned}$$

Moreover, for any $a \in \mathbb{S}^n(\theta_0)$ and all $(x, t) \in \overline{Q}$ with $x \neq 0$, we have

$$|\operatorname{tr}[a(I - \bar{x} \otimes \bar{x})]| \leq \theta_0^{-1}(n-1) < \theta_0^{-1}n \quad \text{and} \quad |\operatorname{tr}[a\bar{x} \otimes \bar{x}]| \leq \theta_0^{-1},$$

and, therefore,

$$\begin{aligned} p_t^\varepsilon - \varepsilon \operatorname{tr}[aD^2p^\varepsilon] - b(x) \cdot Dp^\varepsilon &= h'(r_{x,t})\rho|x|e^{-kt} \left\{ -k - |x|^{-1}b(x) \cdot \bar{x} - \frac{\varepsilon}{|x|^2} \operatorname{tr}[a(I - \bar{x} \otimes \bar{x})] \right\} \\ &\quad - \varepsilon h''(r_{x,t})\rho^2 e^{-2kt} \operatorname{tr}[a\bar{x} \otimes \bar{x}] \\ &\geq h'(r_{x,t})\rho|x|e^{-kt} \left\{ -k + b_0 - \frac{n\varepsilon}{\theta_0|x|^2} \right\} - \varepsilon \|h''\|_{L^\infty} \rho^2 e^{-2kt} \theta_0^{-1}. \end{aligned}$$

Observe that

$$(2.11) \quad \frac{1}{2} \leq r_{x,t} \leq 1 \quad \text{if and only if} \quad \frac{1}{2}R\sqrt{\varepsilon}e^{kt} \leq |x| \leq R\sqrt{\varepsilon}e^{kt},$$

and

$$h'(r_{x,t})\frac{1}{|x|^2} \leq h'(r_{x,t})\frac{4e^{-2kt}}{R^2\varepsilon} \leq h'(r_{x,t})\frac{4}{R^2\varepsilon}.$$

Using the observations above and (1.9) and recalling the choices of the constants and that $a \in \mathbb{S}^n(\theta_0)$ is arbitrary, we get

$$\begin{aligned} p_t^\varepsilon - \varepsilon P^+(D^2p^\varepsilon) - b(x) \cdot Dp^\varepsilon &\geq h'(r_{x,t})\rho|x|e^{-kt} \left\{ -k + b_0 - \frac{4n}{\theta_0 R^2} \right\} - \|h''\|_{L^\infty} \frac{e^{-2kt}}{\theta_0 R^2} \geq -\|h''\|_{L^\infty} \frac{e^{-2kt}}{\theta_0 R^2}. \end{aligned}$$

Thus, noting that, for all $t > 0$,

$$p_t^\varepsilon(0, t) - \varepsilon P^+(D^2 p^\varepsilon(0, t)) - b(0) \cdot D p^\varepsilon(0, t) = 0$$

we conclude that

$$p_t^\varepsilon - \varepsilon P^+(D^2 p^\varepsilon) - b(x) \cdot D p^\varepsilon \geq -\|h''\|_{L^\infty} \frac{e^{-2kt}}{\theta_0 R^2} \quad \text{in } B_{r_0} \times (0, \infty),$$

and, hence, q^ε is a supersolution of (2.3) in $B_{r_0} \times (0, \infty)$.

Finally, we observe that, if $0 \leq t \leq \tau$ and $x \in \partial B_r$, then

$$\frac{|x| e^{-kt}}{\sqrt{\varepsilon} R} \geq \frac{r e^{-k\tau}}{\sqrt{\varepsilon} R} = 1 \quad \text{and} \quad q^\varepsilon(x, t) \geq p^\varepsilon(x, t) = 1,$$

and, if $x \in B_{r/2}$, then

$$\frac{|x| e^{-k\tau}}{\sqrt{\varepsilon} R} \leq \frac{r e^{-k\tau}}{2\sqrt{\varepsilon} R} = \frac{1}{2} \quad \text{and} \quad q^\varepsilon \leq p^\varepsilon + \frac{\|h''\|_{L^\infty}}{b_0 \theta_0 R^2} = \frac{\|h''\|_{L^\infty}}{b_0 \theta_0 R^2}.$$

Moreover,

$$q^\varepsilon(x, 0) = p^\varepsilon(x, 0) = h(|x|/(\sqrt{\varepsilon} R)) \geq \begin{cases} 0 & \text{for all } x \in B_r, \\ 1 & \text{for all } x \in B_r \setminus B_{\sqrt{\varepsilon} R}. \end{cases} \quad \square$$

An application of the Harnack inequality. We use a consequence of the Harnack inequality to obtain an a priori bound for the oscillations of the u^ε 's, which are uniform in ε and t up to ∞ .

If $u^\varepsilon \in C^{2,1}(Q)$ is a solution of (2.1), then

$$v^\varepsilon(y, t) := u^\varepsilon(\sqrt{\varepsilon} y, t) \quad \text{for } (y, t) \in B_{r_0/\sqrt{\varepsilon}} \times [0, \infty),$$

satisfies

$$(2.12) \quad v_t^\varepsilon = \text{tr}[a^\varepsilon(\sqrt{\varepsilon} y, t) D^2 v^\varepsilon] + \frac{b(\sqrt{\varepsilon} y)}{\sqrt{\varepsilon}} \cdot D v^\varepsilon \quad \text{in } B_{r_0/\sqrt{\varepsilon}} \times (0, \infty).$$

It also follows from (1.6) that there exists $L_b > 0$ such that

$$|b(x)| \leq L_b |x| \quad \text{for all } x \in B_{r_0},$$

and, hence,

$$(2.13) \quad \frac{|b(\sqrt{\varepsilon} y)|}{\sqrt{\varepsilon}} \leq L_b |y| \quad \text{for all } y \in B_{r_0/\sqrt{\varepsilon}}.$$

Next we recall the following consequence of the Harnack inequality from Krylov [12, Theorem 4.2.1].

Proposition 5. *Assume (2.2) and (2.13), fix $R \in (0, 2]$, $(z, \tau) \in \mathbb{R}^n \times (0, \infty)$ such that $B_R(z) \subset B_{r_0/\sqrt{\varepsilon}}$ and $\tau > 2R^2$, and let $w \in C^{2,1}(B_R(z) \times (\tau - 2R^2, \tau))$ be a nonnegative solution of (2.12) in $B_R(z) \times (\tau - 2R^2, \tau)$. There exists a constant $C = C(R, \theta_0, L_b, n) > 1$ such that*

$$w(z, \tau - R^2) \leq C \inf_{y \in B_{R/2}(z)} w(y, \tau).$$

We use now Proposition 5 to obtain the following improvement of oscillation-type result for solutions to (2.1).

Corollary 6. Assume (2.2) and (2.13) and, for $\varepsilon \in (0, 1)$, let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (2.1) in Q . Fix $m \in \mathbb{N}$ and $T > 0$ and assume that $(m+2)\sqrt{\varepsilon} \leq r_0$, $T > 4(m+1)$ and

$$(2.14) \quad \begin{cases} u^\varepsilon(0, t) \leq 0 & \text{for all } t \in (0, T), \\ u^\varepsilon(x, t) \leq 1 & \text{for all } (x, t) \in B_{(m+2)\sqrt{\varepsilon}} \times (0, T). \end{cases}$$

There exists a constant $\eta = \eta(m, \theta_0, L_b, n) \in (0, 1)$ such that

$$u^\varepsilon \leq \eta \quad \text{in } B_{m\sqrt{\varepsilon}} \times (4(m+1), T).$$

Proof. Noting that the function $v^\varepsilon(y, t) = u^\varepsilon(\sqrt{\varepsilon}y, t)$ is defined on $B_{m+2} \times (0, T)$, we set

$$w(x, t) = 1 - v^\varepsilon(x, t) \quad \text{for } (x, t) \in B_{m+2} \times (0, T).$$

Observe that w is a solution of (2.12) in $B_{m+2} \times (0, T)$ and, by (2.14), that w is a nonnegative function on $B_{m+2} \times (0, T)$ and satisfies

$$w(0, t) \geq 1 \quad \text{for all } t \in (0, T).$$

Let $(x, t) \in B_m \times (4(m+1), T)$ and choose a finite sequence of balls $B_1(x_1), \dots, B_1(x_m) \subset B_m$ so that $x_1 = 0$, $x \in B_1(x_m)$ and, if $1 \leq i < m$, then $B_1(x_{i+1}) \cap B_1(x_i) \neq \emptyset$. Applying Proposition 5 with $R = 2$ yields, for some $C = C(\theta_0, L_b, n) > 1$,

$$w(0, t - 4m) \leq C \inf_{y \in B_1(x_1)} w(y, t - 4(m-1)),$$

and, hence, if $m = 1$, we have

$$w(0, t - 4m) \leq C^m w(x, t),$$

while, if $m > 1$, repeating the argument above we obtain

$$\begin{aligned} w(0, t - 4m) &\leq C w(x_2, t - 4(m-1)) \leq C^2 \inf_{y \in B_1(x_2)} w(y, t - 4(m-2)) \\ &\leq \dots \leq C^m \inf_{y \in B_1(x_m)} w(y, t) \leq C^m w(x, t). \end{aligned}$$

Thus, we have $w(0, t - 4m) \leq C^m w(x, t)$, and, since $w(0, t - 4m) \geq 1$ by (2.14), we get

$$1 \leq C^m (1 - v^\varepsilon(x, t)),$$

which yields

$$v^\varepsilon(x, t) \leq 1 - \frac{1}{C^m},$$

and, hence, with $\eta = 1 - 1/C^m$,

$$u^\varepsilon(x, t) \leq \eta \quad \text{for all } (x, t) \in B_{m\sqrt{\varepsilon}} \times (4(m+1), T). \quad \square$$

The asymptotic constancy. Let Π be a relatively open, possibly empty, subset of $\partial\Omega$, set $\Omega^\Pi := \Omega \cup \Pi$, and, for any $\delta > 0$,

$$\Omega_\delta := \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) > \delta\} \quad \text{and} \quad \Omega_\delta^\Pi := \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega \setminus \Pi) > \delta\}.$$

The next result is the first indication of what we call asymptotic constancy, which is a straightforward generalization of [11, Theorem 14]. Roughly it says that, for ε small, if a solution of (2.1) is bounded and small (say negative) in a small cylinder around the positive time axis l and a portion of the parabolic boundary, then it is small (of order $\delta > 0$) in a large part of Q after some uniform time depending on δ .

Proposition 7. *Assume (1.3), (1.6), (1.7), (1.8), (1.9) and (2.2) and fix $\delta \in (0, r_0)$. There exist $T_\delta > 0$ and $\varepsilon_0 \in (0, 1)$, which depend only on δ, θ_0, b, Π and Ω , such that, if, for $\varepsilon \in (0, \varepsilon_0)$, $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ is a solution of (2.1) and satisfies, for some $T(\varepsilon) \in (T_\delta, \infty]$,*

$$u^\varepsilon \leq 1 \quad \text{in } \Omega \times [0, T(\varepsilon)) \quad \text{and} \quad u^\varepsilon \leq 0 \quad \text{in } (B_\delta \cup \Pi) \times [0, T(\varepsilon)),$$

then

$$u^\varepsilon(x, t) \leq \delta \quad \text{for all } (x, t) \in \Omega_\delta^\Pi \times [T_\delta, T(\varepsilon)).$$

For the proof of Proposition 7 it is necessary to first describe some preliminary facts that are consequence of the asymptotic stability property of the vector field b .

We fix $\delta > 0$ and set

$$\tau(x) := \sup\{t \geq 0 : X(t, x) \notin B_\delta\} \quad \text{for } x \in \overline{\Omega},$$

where $X(t) = X(t, x)$ is the solution of

$$\dot{X}(t; x) = b(X(t; x)) \quad \text{and} \quad X(0; x) = x.$$

Since Ω is bounded and the origin is a globally asymptotically stable point of b , it is immediate that, if

$$(2.15) \quad T_\delta := \sup_{x \in \overline{\Omega}} \tau(x),$$

then

$$(2.16) \quad 0 < T_\delta < \infty \quad \text{and} \quad X(t, x) \in B_\delta \quad \text{for all } (x, t) \in \overline{\Omega} \times [T_\delta, \infty).$$

We consider the transport problem

$$(2.17) \quad \begin{cases} U_t \leq b \cdot DU & \text{in } \Omega \times (0, T_\delta], \\ \min\{U_t - b \cdot DU, U\} \leq 0 & \text{on } \Pi \times (0, T_\delta], \\ U \leq 0 & \text{in } B_\delta \times \{0\}; \end{cases}$$

the first inequality in (2.17) should be understood in the viscosity subsolution sense while the second is a viscosity interpretation of the Dirichlet condition, $U \leq 0$, on Π (see [10]).

Lemma 2. *Assume (1.3), (1.6), (1.7) and (1.8). If $U \in \text{USC}(\overline{\Omega} \times [0, T_\delta])$ is a subsolution of (2.17), then $U(x, T_\delta) \leq 0$ for all $x \in \Omega^\Pi$.*

Proof. Fix $x \in \Omega^\Pi$ and, for $t \in [0, T_\delta]$, set

$$u(t) = U(X(T_\delta - t, x), t).$$

It is a standard observation (see Lemma A.1 in Appendix A) that $u \in \text{USC}([0, T_\delta])$ is a subsolution, if $x \in \Omega$, of

$$(2.18) \quad u' \leq 0 \quad \text{in } (0, T_\delta],$$

and, if $x \in \Pi$, of

$$(2.19) \quad \begin{cases} u' \leq 0 & \text{in } (0, T_\delta), \\ u' \leq 0 \quad \text{or} \quad u \leq 0 & \text{on } \{T_\delta\}. \end{cases}$$

Suppose that $\max_{[0, T_\delta]} u > 0$. Since $X(T_\delta, x) \in B_\delta$ and $u(0) = U(X(T_\delta, x), 0) \leq 0$, there must exist $\alpha > 0$ and $\tau \in (0, T_\delta]$ such that the function $[0, T_\delta] \ni t \rightarrow u(t) - \alpha t$ attains its maximum on $[0, T_\delta]$ at τ . In view of (2.18), if $x \in \Omega$, then $\alpha \leq 0$, which is a contradiction. If $x \in \Pi$, then either $\alpha \leq 0$ or $\tau = T_\delta$ and $u(T_\delta) \leq 0$, which is again a contradiction. Thus,

we conclude that $u \leq 0$ on $[0, T_\delta]$. In particular, $u(T_\delta) \leq 0$, which shows that $U(x, T_\delta) \leq 0$ for all $x \in \Omega^H$. \square

We proceed with the proof of Proposition 7.

Proof of Proposition 7. Let $T_\delta > 0$ be the number defined by (2.15). For any $\varepsilon \in (0, 1)$, let \mathcal{V}_ε denote the set of all (viscosity) subsolutions $v \in \text{USC}(\overline{\Omega} \times [0, T_\delta])$ of (2.3) such that

$$(2.20) \quad v \leq 1 \text{ on } \overline{\Omega} \times [0, T_\delta] \text{ and } v \leq 0 \text{ on } (B_\delta \cup \Pi) \times [0, T_\delta],$$

and note that \mathcal{V}_ε , which is clearly nonempty, depends only on $\delta, T_\delta, \theta_0, b, \Pi$ and Ω .

It turns out that \mathcal{V}_ε has a maximum element. Indeed, for $(x, t) \in \overline{\Omega} \times [0, T_\delta]$, set

$$v^\varepsilon(x, t) := \sup\{v(x, t) : v \in \mathcal{V}_\varepsilon\}$$

and consider its upper semicontinuous envelope

$$\bar{v}^\varepsilon(x, t) := \limsup_{r \rightarrow 0} \{v^\varepsilon(y, s) : (y, s) \in \overline{\Omega} \times [0, T_\delta], |(y, s) - (x, t)| < r\}.$$

Standard arguments from the theory of viscosity solutions yield that $\bar{v}^\varepsilon \in \mathcal{V}_\varepsilon$ and, since $0 \in \mathcal{V}_\varepsilon$, $\bar{v}^\varepsilon \geq 0$ on $\overline{\Omega} \times [0, T_\delta]$.

Let $U \in \text{USC}(\overline{\Omega} \times [0, T_\delta])$ be the half-relaxed upper limit of \bar{v}^ε , that is, for $(x, t) \in \overline{\Omega} \times [0, T_\delta]$,

$$U(x, t) := \limsup_{\varepsilon \rightarrow 0}^* \bar{v}^\varepsilon(x, t);$$

we refer to Crandall, Ishii and Lions [3] for more discussion about the half relaxed upper and lower limits.

It follows from Lemma 2 that $U(x, T_\delta) \leq 0$ for all $x \in \Omega^H$, and, hence, in view of the uniformity encoded in the definition of U , there exists a constant $\varepsilon_0 \in (0, 1)$, depending only on δ, θ_0, b, Π and Ω , such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$v^\varepsilon(\cdot, T_\delta) \leq \delta \text{ on } \Omega_\delta^H.$$

Finally, since, for each ε , the function $\overline{\Omega} \times [0, T_\delta] \ni (x, t) \mapsto u^\varepsilon(x, s + t)$, with $0 \leq s < T(\varepsilon) - T_\delta$, belongs to \mathcal{V}_ε , it follows that, if $s \in [0, T(\varepsilon) - T_\delta]$, then

$$u^\varepsilon(x, s + T_\delta) \leq v^\varepsilon(x, T_\delta) \leq \delta \quad \text{for all } x \in \Omega_\delta^H \text{ and } \varepsilon \in (0, \varepsilon_0],$$

and, thus,

$$u^\varepsilon(x, t) \leq \delta \quad \text{for all } (x, t) \in \Omega_\delta^H \times [T_\delta, T(\varepsilon)) \text{ and } \varepsilon \in (0, \varepsilon_0]. \quad \square$$

Next we use Corollary 6 and the previous proposition to obtain a refinement. Here we assume an upper bound, say 1, only in a cylindrical neighborhood of the positive time axis l and show that, if, in addition, the solutions are small, say less than 0 on the half line l , then they are small, say less than δ , after a time, of order $|\log \varepsilon|$, in a small cylindrical neighborhood of l . We remark that a time period of order $|\log \varepsilon|$ is “very short” in the logarithmic scale of time, that is, as $\varepsilon \rightarrow 0$, if $\exp(\lambda_\varepsilon/\varepsilon) = O(|\log \varepsilon|)$, then $\lambda_\varepsilon \rightarrow 0$.

Proposition 8. *Assume (1.3), (1.6), (1.7), (1.9) and (2.2). For any $\delta > 0$, there exist $\varepsilon_0 \in (0, 1)$ and a family $\{\tau(\varepsilon)\}_{0 < \varepsilon \leq \varepsilon_0} \subset (0, \infty)$, both depending on r_0, θ_0, b, δ and n , and $\gamma \in (0, 1)$, such that, if, for $\varepsilon \in (0, \varepsilon_0]$, u^ε is a solution of (2.1) with the property that, for some $T(\varepsilon) \in (\tau(\varepsilon), \infty]$,*

$$(2.21) \quad u^\varepsilon \leq 1 \text{ in } B_{r_0} \times (0, T(\varepsilon)) \text{ and } u^\varepsilon(0, t) \leq 0 \text{ for all } t \in (0, T(\varepsilon)),$$

then

$$u^\varepsilon \leq \delta \quad \text{in} \quad B_{\gamma r_0} \times (\tau(\varepsilon), T(\varepsilon)).$$

Moreover, there exists a constant $C > 0$, which depends on r_0, θ_0, b, δ and n , such that

$$\tau(\varepsilon) \leq C(|\log \varepsilon| + 1) \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

Although it appears similar, Proposition 8 is actually very different from [11, Theorem 13]. Indeed the second condition in (2.21) on the solutions is required only at the origin, while in [11, Theorem 13] it is assumed on a neighborhood of the origin. This refinement, which is important for the proofs of Propositions 2, 3 and 4, depends technically on the barrier functions q^ε in Lemma 1 and the Harnack inequality (Proposition 5).

Proof of Proposition 8. To simplify the argument, we assume that $T(\varepsilon) = \infty$ since the general case can be treated similarly.

Fix $\delta > 0$, choose $h \in C^2([0, \infty))$ satisfying (2.6) and $m = m(\theta_0, n, \|h''\|_{L^\infty}) \in \mathbb{N}$ such that

$$\frac{\|h''\|_{L^\infty}}{b_0 \theta_0 m^2} \leq \frac{1}{2} \quad \text{and} \quad m \geq \frac{2\sqrt{2n}}{\sqrt{b_0 \theta_0}},$$

let $\eta = \eta(\theta_0, L_b, n) \in (0, 1)$ be the constant in Corollary 6, set $\tau_0 = 4(m+1)$ and fix $\varepsilon_1 = \varepsilon_1(r_0, m) \in (0, 1)$ so that

$$(m+2)\sqrt{\varepsilon_1} \leq r_0.$$

Then, for any $\varepsilon \in (0, \varepsilon_1]$, Corollary 6 gives

$$u^\varepsilon(x, t) \leq \eta \quad \text{for all } (x, t) \in B_{m\sqrt{\varepsilon}} \times (\tau_0, \infty).$$

Define

$$v^\varepsilon := (1 - \eta)^{-1} (u^\varepsilon - \eta) \quad \text{in } \Omega \times [0, \infty),$$

and note that v^ε is a solution of (2.1), and, moreover,

$$v^\varepsilon \leq 1 \quad \text{in } B_{r_0} \times (0, \infty) \quad \text{and} \quad v^\varepsilon \leq 0 \quad \text{on } \overline{B}_{m\sqrt{\varepsilon}} \times [\tau_0, \infty).$$

Let q^ε be given by (2.10) with R and r replaced by m and r_0 respectively. It follows from Lemma 1 and the comparison principle that, for any fixed $s \geq \tau_0$,

$$v^\varepsilon(\cdot, s + \cdot) \leq q^\varepsilon \quad \text{in } B_{r_0} \times [0, \tau_1],$$

where $\tau_1 = \tau_1(\varepsilon) > 0$ is given by

$$\frac{\theta_0 \tau_1}{2} = \log \left(\frac{r_0}{m\sqrt{\varepsilon}} \right).$$

Hence,

$$v^\varepsilon(\cdot, \cdot + \tau_1) \leq \frac{\|h''\|_{L^\infty}}{b_0 \theta_0 m^2} \leq \frac{1}{2} \quad \text{in } B_{r_0/2} \times [\tau_0, \infty),$$

which, with $T_1(\varepsilon) := \tau_0 + \tau_1(\varepsilon)$, can be rewritten as

$$(2.22) \quad u^\varepsilon \leq \eta + \frac{1 - \eta}{2} = \frac{1}{2}(1 + \eta) \quad \text{in } B_{r_0/2} \times [T_1(\varepsilon), \infty).$$

Next, for $j = 2, 3, \dots$, we choose $\varepsilon_j \in (0, \varepsilon_{j-1})$ so that

$$(m+2)\sqrt{\varepsilon_j} \leq \frac{r_0}{2^{j-1}},$$

and, for any $\varepsilon \in (0, \varepsilon_j)$, select $\tau_j = \tau_j(\varepsilon) > \tau_{j-1}(\varepsilon)$ so that

$$\frac{\theta_0 \tau_j(\varepsilon)}{2} = \log \left(\frac{r_0}{2^{j-1} m \sqrt{\varepsilon}} \right),$$

and set, for $\varepsilon \in (0, \varepsilon_j)$,

$$T_j(\varepsilon) := T_{j-1}(\varepsilon) + \tau_0 + \tau_j(\varepsilon) = j\tau_0 + \sum_{i=1}^j \tau_i(\varepsilon).$$

We prove by induction that

$$(2.23) \quad u^\varepsilon \leq \left(\frac{1+\eta}{2} \right)^j \quad \text{in } B_{r_0/2^j} \times [T_j(\varepsilon), \infty).$$

Since (2.22) yields that (2.23) holds for $j = 1$, we assume that (2.23) is valid for some $j \in \mathbb{N}$, set

$$w^\varepsilon := \left(\frac{2}{1+\eta} \right)^j u^\varepsilon(\cdot, \cdot + T_j(\varepsilon)) \quad \text{in } Q,$$

observe that w^ε is a solution of (2.1), with $a^\varepsilon(\cdot, \cdot)$ replaced by $a^\varepsilon(\cdot, \cdot + T_j(\varepsilon))$ and satisfies $w^\varepsilon(0, t) \leq 0$ for all $t \in [0, \infty)$ and $w^\varepsilon \leq 1$ in $B_{r_0/2^j} \times [0, \infty)$.

Using Lemma 1 and Corollary 6 as before, with the same m and τ_0 , but with u^ε , r_0 and τ_1 replaced by w^ε , $r_0/2^j$ and τ_{j+1} respectively, we obtain

$$w^\varepsilon \leq \frac{1+\eta}{2} \quad \text{in } B_{r_0/2^{j+1}} \times (\tau_0 + \tau_{j+1}(\varepsilon), \infty),$$

which, after been rewritten as

$$u^\varepsilon \leq \left(\frac{1+\eta}{2} \right)^{j+1} \quad \text{in } B_{r_0/2^{j+1}} \times [T_{j+1}(\varepsilon), \infty),$$

yields the claim.

Finally, selecting $j \in \mathbb{N}$ so that

$$\left(\frac{1+\eta}{2} \right)^j \leq \delta,$$

setting $\varepsilon_0 = \varepsilon_j$, $\gamma = 2^{-j}$ and $\tau(\varepsilon) = T_j(\varepsilon)$, and observing that, as $\varepsilon \rightarrow 0+$, $\tau(\varepsilon) = O(|\log \varepsilon|)$ we complete the proof. \square

We have by now completed all the technical steps needed for the next theorem, which is a nontrivial refinement of Proposition 7. It asserts that bounded solutions to (2.1), which are small on the positive time axis l and a part of the parabolic boundary, are actually small in almost the whole domain after some time of order $|\log \varepsilon|$. This is the mathematical statement of what we called asymptotic constancy.

Theorem 9. *Assume (1.3), (1.6), (1.7), (1.8), (1.9) and (2.2) and let $\{T(\varepsilon)\}_{\varepsilon \in (0,1)}$ be a collection of positive numbers. For each $\delta > 0$ and $C_0 > 0$, there exist constants $\varepsilon_0 \in (0, 1)$ and $C > 0$ such that, if, for $\varepsilon \in (0, \varepsilon_0]$, $u^\varepsilon \in C^{2,1}(Q)$ is a solution of (2.1) satisfying*

$$u^\varepsilon \leq C_0 \quad \text{in } \Omega \times [0, T(\varepsilon)) \quad \text{and} \quad u^\varepsilon \leq 0 \quad \text{in } (\{0\} \cup \Pi) \times [0, T(\varepsilon)),$$

then

$$u^\varepsilon(x, t) \leq \delta \quad \text{for all } (x, t) \in \Omega_\delta^\Pi \times (C|\log \varepsilon|, T(\varepsilon)).$$

Proof. Proposition 8 yields constants $\varepsilon_1, \gamma \in (0, 1)$ and $C_1 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_1$,

$$u^\varepsilon \leq \frac{\delta}{2} \quad \text{in } B_{\gamma r_0} \times [C_1 |\log \varepsilon|, T(\varepsilon)).$$

Proposition 7 applied to $v^\varepsilon(x, t) := C_0^{-1}(u^\varepsilon(x, t + C_1 |\log \varepsilon|) - \delta)$ instead u^ε implies the existence of T_δ and $\varepsilon_0 \in (0, \varepsilon_1)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$(2.24) \quad v^\varepsilon \leq \frac{\delta}{2C_0} \quad \text{in } \Omega_\delta^\Pi \times [T_\delta, T(\varepsilon) - C_1 |\log \varepsilon|),$$

which says that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$u^\varepsilon \leq \delta \quad \text{in } \Omega_\delta^\Pi \times [T_\delta + C_1 |\log \varepsilon|, T(\varepsilon)),$$

and the proof is complete. \square

Next we use the last result to control the difference between values of $u^\varepsilon(\cdot, t)$ and $u^\varepsilon(0, t)$.

Theorem 10. *Assume (1.3), (1.6), (1.7), (1.8), (1.9) and (2.2). For each $\delta > 0$ and $C_0 > 0$ there exist constants $\varepsilon_0 \in (0, 1)$ and $C > 0$ such that, if, for $\varepsilon \in (0, \varepsilon_0]$, u^ε is a solution of (2.1) satisfying*

$$|u^\varepsilon| \leq C_0 \quad \text{in } \Omega \times [0, \infty),$$

then

$$|u^\varepsilon(x, t) - u^\varepsilon(0, t)| \leq \delta \quad \text{for all } (x, t) \in \Omega_\delta \times [C |\log \varepsilon|, \infty).$$

Proof. We double the variables and define the function $v^\varepsilon : \Omega \times \Omega \times [0, \infty) \rightarrow \mathbb{R}$ by

$$v^\varepsilon(x, y, t) := u^\varepsilon(x, t) - u^\varepsilon(y, t).$$

It is standard that v^ε solves in $\Omega \times \Omega \times (0, \infty)$ the doubled equation

$$\begin{aligned} v_t^\varepsilon &= \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D_x^2 v^\varepsilon] + \varepsilon \operatorname{tr}[a^\varepsilon(y, t) D_y^2 v^\varepsilon] + b(x) \cdot D_x v^\varepsilon + b(y) \cdot D_y v^\varepsilon \\ &= \varepsilon \operatorname{tr}[A^\varepsilon(x, y, t) D^2 v^\varepsilon] + B(x, y) \cdot D v^\varepsilon, \end{aligned}$$

where

$$B(x, y) := (b(x), b(y)) \quad \text{and} \quad A^\varepsilon(x, y, t) := \begin{pmatrix} a^\varepsilon(x, t) & 0 \\ 0 & a^\varepsilon(y, t) \end{pmatrix}.$$

The conclusion follows if we apply Theorem 9, with $\Pi = \emptyset$, to $\pm v^\varepsilon$, since $v^\varepsilon(0, 0, t) = 0$ for all $t \geq 0$ and $|v^\varepsilon| \leq 2C_0$ in $\Omega \times \Omega \times [0, \infty)$.

The only issue is that the boundary of $\Omega \times \Omega$ does not have the C^1 -regularity required for the theorem.

To overcome this difficulty, we only need to approximate $\Omega \times \Omega$ by smaller C^1 -domains, that is, for fixed $\delta > 0$, we choose a C^1 -domain $W \subset \mathbb{R}^{2n}$ so that

$$\Omega_\delta \times \Omega_\delta \subset W_{\delta/2} \subset W \subset \Omega \times \Omega,$$

where $W_{\delta/2} := \{(x, y) \in W : \operatorname{dist}((x, y), \partial W) > \delta/2\}$, and

$$B(x, y) \cdot N(x, y) < 0 \quad \text{for all } (x, y) \in \partial W,$$

where $N(x, y)$ denotes the outward unit normal vector at $(x, y) \in \partial W$. \square

3. QUASI-POTENTIALS

We establish here an important continuity property under perturbations for the minimum and the arg min map of the quasi-potentials we introduced earlier in the introduction.

We begin with some notation and the introduction of several auxiliary quantities needed to define the perturbations. To this end, we fix $\beta_0 \in I_g$, define $H_0 \in C(\overline{\Omega} \times \mathbb{R}^n)$ by

$$H_0(x, p) = a(x, \beta_0)p \cdot p + b(x) \cdot p,$$

choose some $\delta_0 > 0$, and, for $\delta \in (0, \delta_0)$,

$$\theta(\delta) := \max\{|(a(x, c) - a(x, \beta_0))\xi \cdot \xi| : x \in \overline{\Omega}, \xi \in \mathbb{R}^n, |\xi| \leq 1, c \in [\beta_0 - \delta, \beta_0 + \delta]\}.$$

The continuity of $a(x, c)$ (recall (1.5)) yields $\lim_{\delta \rightarrow 0} \theta(\delta) = 0$, and, hence, selecting $\delta_0 > 0$ sufficiently small, we assume henceforth that

$$\theta(\delta) \leq \theta_0/2 \quad \text{for all } \delta \in (0, \delta_0).$$

We define $a_\delta^\pm \in C(\overline{\Omega}, \mathbb{S}^n)$ and $H_\delta^\pm \in C(\overline{\Omega} \times \mathbb{R}^n)$ respectively by

$$a_\delta^\pm(x) := a(x, \beta_0) \pm \theta(\delta)I \quad \text{and} \quad H_\delta^\pm(x, p) := a_\delta^\pm(x)p \cdot p + b(x) \cdot p,$$

and note that, for all $(x, c) \in \overline{\Omega} \times [\beta_0 - \delta, \beta_0 + \delta]$,

$$(\theta_0/2)I \leq a_\delta^-(x) \leq a(x, c) \leq a_\delta^+(x) \leq (\theta_0^{-1} + \theta_0/2)I.$$

We choose $\chi_\delta \in C(\mathbb{R}^n; [0, 1])$ such that

$$\chi_\delta = 1 \quad \text{in} \quad x \in \Omega_\delta \quad \text{and} \quad \chi_\delta = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega_{\delta/2},$$

and define $\mathcal{H}_\delta^\pm \in C(\overline{\Omega} \times \mathbb{R}^n)$ by

$$\mathcal{H}_\delta^+(x, p) = \chi_\delta(x)H_\delta^+(x, p) + (1 - \chi_\delta(x))(\theta_0^{-1}|p|^2 + b(x) \cdot p),$$

$$\mathcal{H}_\delta^-(x, p) = \chi_\delta(x)H_\delta^-(x, p) + (1 - \chi_\delta(x))(\theta_0|p|^2 + b(x) \cdot p),$$

and note that, for all $(x, c) \in \Omega_{\delta/2} \times [\beta_0 - \delta, \beta_0 + \delta] \cup (\Omega \setminus \Omega_{\delta/2}) \times \mathbb{R}$ and $p \in \mathbb{R}^n$,

$$\mathcal{H}_\delta^-(x, p) \leq a(x, c)p \cdot p + b(x) \cdot p \leq \mathcal{H}_\delta^+(x, p).$$

We also have

$$\mathcal{H}_\delta^\pm(x, p) = H_\delta^\pm(x, p) \quad \text{for all } (x, p) \in \Omega_\delta \times \mathbb{R}^n,$$

while, for all $(x, p) \in (\overline{\Omega} \setminus \Omega_{\delta/2}) \times \mathbb{R}^n$,

$$\mathcal{H}_\delta^+(x, p) = \theta_0^{-1}|p|^2 + b(x) \cdot p \quad \text{and} \quad \mathcal{H}_\delta^-(x, p) = \theta_0|p|^2 + b(x) \cdot p.$$

If we set

$$\alpha_\delta^+(x) = \chi_\delta(x)a_\delta^+(x) + (1 - \chi_\delta(x))\theta_0^{-1}I \quad \text{and} \quad \alpha_\delta^-(x) = \chi_\delta(x)a_\delta^-(x) + (1 - \chi_\delta(x))\theta_0I,$$

then, for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$,

$$\mathcal{H}_\delta^\pm(x, p) = \alpha_\delta^\pm(x)p \cdot p + b(x) \cdot p.$$

Let V_0 and V_δ^\pm be respectively the maximal subsolutions of

$$(3.1) \quad \begin{cases} H_0(x, Du) = 0 & \text{in } \Omega, \\ u(0) = 0, \end{cases}$$

and

$$(3.2) \quad \begin{cases} \mathcal{H}_\delta^\pm(x, Du) = 0 & \text{in } \Omega, \\ u(0) = 0. \end{cases}$$

We note by [11, Corollary 5] that $V_\delta^\pm(x) > 0$ and $V_0(x) > 0$ for all $x \in \overline{\Omega} \setminus \{0\}$. Since $\mathcal{H}_\delta^- \leq H_0 \leq \mathcal{H}_\delta^+$ on $\Omega \times \mathbb{R}^n$, it is clear that

$$(3.3) \quad V_\delta^+ \leq V_0 \leq V_\delta^- \quad \text{on } \overline{\Omega}.$$

We set

$$M_0 := \min_{\partial\Omega} V_0, \quad \Gamma_0 := \arg \min(V_0|\partial\Omega), \quad M_\delta^\pm := \min_{\partial\Omega} V_\delta^\pm, \quad \Gamma_\delta^\pm := \arg \min(V_\delta^\pm|\partial\Omega),$$

and note that

$$M_\delta^+ \leq M_0 \leq M_\delta^-.$$

The following result is about the continuity of M_δ^\pm and Γ_δ^\pm with respect to δ .

Proposition 11. *Assume (1.3), (1.5), (1.6), (1.7) and (1.8). Then*

$$(3.4) \quad \lim_{\delta \rightarrow 0^+} M_\delta^+ = \lim_{\delta \rightarrow 0^+} M_\delta^- = M_0$$

and

$$(3.5) \quad \limsup_{\delta \rightarrow 0^+} \Gamma_\delta^+ \cup \limsup_{\delta \rightarrow 0^+} \Gamma_\delta^- \subset \Gamma_0.$$

The set limit in (3.5) is understood in the sense of Kuratowski, that is, for a given $\{\Gamma_\delta\}_{\delta \in (0, \delta_0)} \subset \mathbb{R}^n$,

$$\limsup_{\delta \rightarrow 0^+} \Gamma_\delta := \bigcap_{r \in (0, \delta_0)} \overline{\bigcup_{\delta \in (0, r)} \Gamma_\delta} = \{x \in \mathbb{R}^n : x = \lim_{k \rightarrow \infty} x_k, x_k \in \Gamma_{\delta_k}, \lim_{k \rightarrow \infty} \delta_k = 0\}.$$

Now we prove Proposition 11.

Proof. The uniform in x and δ coercivity of the Hamiltonians \mathcal{H}_δ^\pm , that is the fact that $\mathcal{H}_\delta^\pm(x, p) \rightarrow \infty$ as $|p| \rightarrow \infty$ uniformly in x and δ , yields that the families $\{V_\delta^\pm\}_{\delta \in (0, \delta_0)}$ are equi-Lipschitz continuous on $\overline{\Omega}$, and, since $V_\delta^\pm(0) = 0$, relatively compact in $C(\overline{\Omega})$.

To prove (3.4) and (3.5), it is enough to show that, if $\{\delta_j\}_{j \in \mathbb{N}} \subset (0, \delta_0)$ is such that both $\{V_{\delta_j}^\pm\}_{j \in \mathbb{N}}$ converge in $C(\overline{\Omega})$ to some $V_0^\pm \in C(\overline{\Omega})$, that is

$$V_0^\pm = \lim_{j \rightarrow \infty} V_{\delta_j}^\pm \quad \text{uniformly on } \overline{\Omega},$$

then

$$(3.6) \quad M_0 = \min_{\partial\Omega} V_0^+ = \min_{\partial\Omega} V_0^-.$$

and

$$(3.7) \quad \arg \min(V_0|\partial\Omega) = \arg \min(V_0^+|\partial\Omega) = \arg \min(V_0^-|\partial\Omega).$$

For notational convenience, we set

$$M_0^\pm := \min_{\partial\Omega} V_0^\pm \quad \text{and} \quad \Gamma_0^\pm = \arg \min(V_0^\pm|\partial\Omega).$$

It is well-known (see Lemma B.1 in the Appendix) that the V_δ^\pm 's satisfy in the viscosity sense

$$\mathcal{H}_\delta^\pm(x, DV_\delta^\pm) \geq 0 \quad \text{on } \overline{\Omega} \quad \text{and} \quad \mathcal{H}_\delta^\pm(x, DV_\delta^\pm) \leq 0 \quad \text{in } \Omega,$$

that is, the V_δ^\pm 's are solutions of the state-constraints problems

$$\mathcal{H}_\delta^\pm(x, DV_\delta^\pm) = 0 \quad \text{in } \Omega.$$

By the stability of viscosity properties, the V_0^\pm 's satisfy

$$H_0(x, DV_0^\pm(x)) \leq 0 \quad \text{in } \Omega \quad \text{and} \quad H_{\theta_0}^+(x, DV_0^+(x)) \geq 0 \quad \text{on } \overline{\Omega},$$

where

$$H_{\theta_0}^+(x, p) := \begin{cases} H_0(x, p) & \text{for } (x, p) \in \Omega \times \mathbb{R}^n, \\ \theta_0^{-1}|p|^2 + b(x) \cdot p & \text{for } (x, p) \in \partial\Omega \times \mathbb{R}^n. \end{cases}$$

Here we used that

$$\limsup_{\delta \rightarrow 0}^* \mathcal{H}_\delta^\pm(x, p) = \liminf_{\delta \rightarrow 0}^* \mathcal{H}_\delta^\pm(x, p) = H_0(x, p) \quad \text{for all } (x, p) \in \Omega \times \mathbb{R}^n,$$

and

$$\limsup_{\delta \rightarrow 0}^* \mathcal{H}_\delta^+(x, p) = H_{\theta_0}^+(x, p) \quad \text{for all } (x, p) \in \overline{\Omega} \times \mathbb{R}^n.$$

The maximality of V_0 implies that $V_0^- \leq V_0$ on Ω and, since, in view of (3.3), $V_0 \leq V_0^-$ in $\overline{\Omega}$, we have $V_0^- = V_0$, which, obviously gives

$$(3.8) \quad M_0 = M_0^- \quad \text{and} \quad \Gamma_0^- = \Gamma_0.$$

The argument for M_0^+ and Γ_0^+ is slightly more complicated.

Since (3.3) yields $V_0^+ \leq V_0$, it is immediate that

$$M_0^+ \leq M_0.$$

Next we show that

$$(3.9) \quad \min\{V_0, M_0\} \leq V_0^+ \quad \text{in } \overline{\Omega},$$

which, together the previous inequality, gives

$$(3.10) \quad M_0^+ = M_0 \quad \text{and} \quad \Gamma_0 \subset \Gamma_0^+.$$

We proceed with the proof of (3.9). Fix $l \in (0, M_0)$, choose $\gamma_1 \in (0, \delta_0)$ so that

$$V_0 > l \quad \text{on } \overline{\Omega} \setminus \Omega_{\gamma_1},$$

fix $\mu \in (0, 1)$ sufficiently close to 1 so that

$$\mu V_0 > l \quad \text{on } \overline{\Omega} \setminus \Omega_{\gamma_1},$$

and choose $\gamma_2 \in (0, \gamma_1)$ so that

$$\mu(a(x, \beta_0) + \theta(\delta)I) \leq a(x, \beta_0) \quad \text{for all } x \in \overline{\Omega} \quad \text{and } \delta \in (0, \gamma_2).$$

Observe that, if $u_\mu(x) := \mu V_0(x)$, then, for all $\delta \in (0, \gamma_2)$,

$$u_\mu > l \quad \text{in } \overline{\Omega} \setminus \Omega_\delta,$$

and, for all $\delta \in (0, \gamma_2)$, in the viscosity sense,

$$\begin{aligned} H_\delta^+(x, Du_\mu) &= \mu(\mu(a(x, \beta_0) + \theta(\delta)I)DV_0 \cdot DV_0 + b(x) \cdot DV_0) \\ &\leq \mu(a(x, \beta_0)DV_0 \cdot DV_0 + b \cdot DV_0) \leq \mu H_0(x, DV_0) \leq 0 \quad \text{in } \Omega. \end{aligned}$$

Now set $u_\mu^l := \min\{u_\mu, l\}$ and note that the convexity of $H_\delta^+(x, p)$ in p yields that, if $\delta \in (0, \gamma_2)$, then

$$\mathcal{H}_\delta^+(x, Du_\mu^l) = H_\delta^+(x, Du_\mu^l) \leq 0 \quad \text{in } \Omega_\delta.$$

Also, if $\delta \in (0, \gamma_2)$, then, since $u_\mu^l(x) = l$ in an open neighborhood $N_\delta \subset \Omega$ of $\Omega \setminus \Omega_\delta$,

$$\mathcal{H}_\delta(x, Du_\mu^l) = 0 \quad \text{in } N_\delta.$$

Thus we deduce that, for any $\delta \in (0, \gamma_2)$, u_μ^l is a subsolution of $\mathcal{H}_\delta^+(x, Du_\mu^l) \leq 0$ in Ω , and, hence, $u_\mu^l \leq V_\delta^+$ in $\overline{\Omega}$ by the maximality of V_δ^+ . Sending $\delta \rightarrow 0$, along the sequence $\{\delta_j\}$, $\mu \rightarrow 1$ and $l \rightarrow M_0$ in this order, we conclude that (3.9) holds.

Next we show that $\Gamma_0^+ \subset \Gamma_0$. Let $z \in \Gamma_0^+ \setminus \Gamma_0$ and observe that, since $V_0(z) > M_0$, there is an open, relatively to $\overline{\Omega}$, neighborhood $N_z \subset \overline{\Omega}$, such that $V_0 > M_0$ in N_z , while (3.9) gives $V_0^+ \geq M_0$ in N_z .

Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω , that is, $\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}$ and $|D\rho| \neq 0$ on $\partial\Omega$, and, in particular, $D\rho/|D\rho| = \nu$ on $\partial\Omega$.

For any $\varepsilon > 0$, $x \mapsto V_0^+(x) - \varepsilon\rho(x)$ achieves a minimum at z over N_z . Since $H_{\theta_0}^+(x, DV_0^+) \geq 0$ on $\overline{\Omega}$, we have

$$0 \leq H_{\theta_0}^+(z, \varepsilon D\rho(z)) = \varepsilon(\varepsilon\theta_0^{-1}|D\rho(z)|^2 + b(z) \cdot D\rho(z)),$$

which is a contradiction, in view of the fact that the right hand side is negative if ε is sufficiently small.

It follows that $\Gamma_0^+ \setminus \Gamma_0 = \emptyset$, that is, $\Gamma_0^+ \subset \Gamma_0$, which, together with (3.10), proves the claim. \square

4. BARRIER FUNCTIONS

We adapt and modify here the main argument of building barrier functions of [11] to obtain information on the behavior of the solutions u^ε of (2.1) along the positive time axis l , that is on $u^\varepsilon(0, t)$, for a sufficiently long time interval $[0, T(\varepsilon))$, under the assumption that the matrices $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)})$ are bounded by $\alpha \in C(\overline{Q}_{T(\varepsilon)})$ from above or from below.

Recall that, for any $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $H_\alpha \in C(\overline{\Omega} \times \mathbb{R}^n)$ is the Hamiltonian given by $H_\alpha(x, p) = \alpha(x)p \cdot p + b(x) \cdot p$, $V_\alpha \in \text{Lip}(\overline{\Omega})$ is the quasi-potential corresponding to (α, b) , and $M_\alpha = \min_{\partial\Omega} V_\alpha$, and set

$$\Sigma_\alpha := \{x \in \overline{\Omega} : V_\alpha(x) \leq M_\alpha\}, \quad \Gamma_\alpha := \Sigma_\alpha \cap \partial\Omega,$$

and, for any $m > 0$,

$$\Sigma_\alpha^m := \{x \in \overline{\Omega} : V_\alpha(x) \leq m\}.$$

We consider again (2.1) for a family of $a^\varepsilon \in C(\overline{Q}, \mathbb{S}^n(\theta_0))$ and present two results, one stated in the form of an upper bound and the other in the form of a lower bound. The upper bound is valid up to λ smaller than M_α in the logarithmic time scale, and the lower bound is valid up to ∞ , provided u^ε , on the boundary portion $\Gamma_\alpha \times [0, T(\varepsilon))$, is larger than a specified lower bound.

We begin with the former, which corresponds to [11, Theorem 1 (i)] in its nature. The latter is related to [11, Theorem 1(ii)].

Proposition 12. *Assume (1.10) and fix $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $T(\varepsilon) \in (0, \infty]$, $C_0 > 0$ and $m \in (0, M_\alpha)$. If, for $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}; \mathbb{S}^n(\theta_0))$ such that $a^\varepsilon \leq \alpha$ in $Q_{T(\varepsilon)}$, $u^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}) \cap C^{2,1}(Q_{T(\varepsilon)})$ is a subsolution of (2.1) in $Q_{T(\varepsilon)}$ such that*

$$u^\varepsilon(\cdot, 0) \leq 0 \quad \text{in } \Sigma_\alpha^m \quad \text{and} \quad u^\varepsilon \leq C_0 \quad \text{in } Q_{T(\varepsilon)},$$

then, for any $\delta > 0$, there exists $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$u^\varepsilon(0, t) \leq \delta \quad \text{for all } t \in [0, \exp((m - \delta)/\varepsilon) \wedge T(\varepsilon)).$$

The lower bound is stated next.

Proposition 13. Assume (1.10), fix $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $T(\varepsilon) \in (0, \infty]$, $C_0 > 0$ and $m > M_\alpha$. If, for $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}; \mathbb{S}^n(\theta_0))$ such that $a^\varepsilon \geq \alpha$ in $Q_{T(\varepsilon)}$, $u^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}) \cap C^{2,1}(Q_{T(\varepsilon)})$ is a supersolution of (2.1) in $Q_{T(\varepsilon)}$ such that

$$u^\varepsilon(\cdot, 0) \geq 0 \quad \text{in } \Sigma_\alpha^m, \quad u^\varepsilon \geq 0 \quad \text{in } (\Sigma_\alpha^m \cap \partial\Omega) \times (0, T(\varepsilon)) \quad \text{and } u^\varepsilon \geq -C_0 \quad \text{in } Q_{T(\varepsilon)},$$

then, for any $\delta > 0$, there exists $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$u^\varepsilon(0, t) \geq -\delta \quad \text{for all } t \in [0, T(\varepsilon)).$$

The proofs of Propositions 12 and 13 use the next two lemmata; for their proof we refer to [11].

Lemma 3. Assume (1.10) and fix $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$. For any $r \in (0, r_0)$, there exist $v_r \in C^2(\overline{\Omega})$ and $\eta \in (0, 1)$ such that

$$(4.1) \quad \begin{cases} H_\alpha(x, Dv_r) \leq -\eta & \text{in } \Omega \setminus B_r, \\ H_\alpha(x, Dv_r) \leq 1 & \text{in } B_r, \\ \|v_r - V_\alpha\|_{L^\infty(\Omega)} < r. \end{cases}$$

Lemma 4. Assume (1.10) and fix $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$. For each $m > M_\alpha$, there exists $w_m \in \text{Lip}(\overline{\Omega})$ and $\eta > 0$ such that

$$(4.2) \quad 0 < \min_{\overline{\Omega}} w_m \leq \max_{\overline{\Omega}} w_m < m,$$

and, in the viscosity supersolution sense,

$$(4.3) \quad H_\alpha(x, -Dw_m) \geq \eta \quad \text{in } \Omega \quad \text{and} \quad D^2w_m(x) \leq \eta^{-1}I \quad \text{in } \Omega.$$

We continue with the proof of Proposition 12 which parallels that of [11, Theorem 8].

Proof of Proposition 12. For $r \in (0, r_0)$ to be fixed below, let $v = v_r \in C^2(\overline{\Omega})$ (for notational simplicity we omit the subscript r in what follows) and $\eta > 0$ be given by Lemma 3, set, for $x \in \overline{\Omega}$,

$$w^\varepsilon(x) := \exp\left(\frac{v(x) - m + 2r}{\varepsilon}\right),$$

compute, for any $(x, t) \in Q$,

$$\begin{aligned} & \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 w^\varepsilon] + b(x) \cdot Dw^\varepsilon \\ &= \frac{w^\varepsilon}{\varepsilon} (a^\varepsilon(x, t) Dv \cdot Dv + b \cdot Dv + \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 v]) \\ &\leq \frac{w^\varepsilon}{\varepsilon} (\alpha(x) Dv \cdot Dv + b(x) \cdot Dv + \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 v]) \\ &\leq \frac{w^\varepsilon}{\varepsilon} (H_\alpha(x, Dv) + \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 v]). \end{aligned}$$

and choose $\varepsilon_0 \in (0, 1)$ so that

$$\varepsilon_0 (\operatorname{tr} a^\varepsilon(x, t) D^2 v)_+ \leq \min\{\eta, r, 1\};$$

note that ε_0 can be chosen so as to depend on a^ε only through θ_0 .

We assume henceforth that $\varepsilon \in (0, \varepsilon_0)$ and observe that, from the computation above, we get

$$(4.4) \quad \varepsilon \operatorname{tr}[a^\varepsilon(x, t) D^2 w^\varepsilon] + b(x) \cdot Dw^\varepsilon \leq \begin{cases} 0 & \text{for all } (x, t) \in \Omega \setminus B_r \times (0, \infty), \\ \frac{2}{\varepsilon} w^\varepsilon & \text{for all } (x, t) \in B_r \times (0, \infty). \end{cases}$$

Let $C_0 > 0$ be a Lipschitz bound of b , and note that, if $H_\alpha(x, p) \leq 0$, then $|p| \leq C_0 \theta_0^{-1}$, which implies that $V_\alpha(x) \leq C_0 |x|^2 / (2\theta_0) \leq C_0 r^2 / (2\theta_0)$ for all $x \in B_r$. We may thus assume by replacing, if needed, $r > 0$ by a smaller number that $V_\alpha \leq r$ in B_r . Accordingly we have

$$v - m + 2r \leq V_\alpha - m + 3r \leq -m + 4r \quad \text{in } B_r,$$

and

$$(4.5) \quad w^\varepsilon \leq \exp\left(\frac{-m + 4r}{\varepsilon}\right) \quad \text{in } B_r.$$

Observe also that

$$v - m + 2r > V_\alpha - m + r \geq r \quad \text{in } \overline{\Omega} \setminus \Sigma_\alpha^m,$$

and

$$(4.6) \quad w^\varepsilon > \exp\left(\frac{r}{\varepsilon}\right) \quad \text{in } \overline{\Omega} \setminus \Sigma_\alpha^m.$$

Next set $d_\varepsilon = \frac{2}{\varepsilon} \exp(\frac{-m+4r}{\varepsilon})$ and

$$z^\varepsilon(x, t) = w^\varepsilon(x) + d_\varepsilon t \quad \text{for } (x, t) \in \overline{\Omega} \times [0, \infty).$$

It is immediate from (4.4) and (4.5) that

$$(4.7) \quad z_t^\varepsilon \geq \varepsilon \operatorname{tr}[a^\varepsilon D^2 z^\varepsilon] + b \cdot Dz^\varepsilon \quad \text{in } Q.$$

We choose $C_1 > 0$ so that, for all $\varepsilon \in (0, 1)$,

$$u^\varepsilon \leq C_1 \quad \text{on } \overline{Q},$$

and by replacing, if necessary, $\varepsilon_0 > 0$ by a smaller number we may assume that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$C_1 < \exp\left(\frac{r}{\varepsilon}\right).$$

It follows from (4.6) that

$$z^\varepsilon \geq w^\varepsilon \geq \exp\left(\frac{r}{\varepsilon}\right) > C_1 \geq u^\varepsilon \quad \text{on } (\overline{\Omega} \setminus \Sigma_\alpha^m) \times [0, \infty);$$

note that, since $m < M_\alpha$, we have $\partial\Omega \subset \overline{\Omega} \setminus \Sigma_\alpha^m$.

On the other hand, for any $x \in \Sigma_\alpha^m$, we have

$$z^\varepsilon(x, 0) = w^\varepsilon(x) > 0 \geq u^\varepsilon(x, 0),$$

and, hence,

$$u^\varepsilon \leq z^\varepsilon \quad \text{on } \partial_p Q.$$

We find from the above, (4.7) and the comparison principle that

$$u^\varepsilon \leq z^\varepsilon \quad \text{on } \overline{Q},$$

and, in particular, for any $t \in [0, \exp((m - 5r)/\varepsilon)]$,

$$u^\varepsilon(0, t) \leq z^\varepsilon(0, t) \leq w^\varepsilon(0) + \frac{2}{\varepsilon} \exp\left(\frac{-r}{\varepsilon}\right) \leq \exp\left(\frac{-m + 3r}{\varepsilon}\right) + \frac{2}{\varepsilon} \exp\left(\frac{-r}{\varepsilon}\right).$$

It is now clear that, for a given $\delta > 0$, we may choose $r > 0$ and $\varepsilon_0 \in (0, 1)$ so that if $\varepsilon \in (0, \varepsilon_0)$, then

$$u^\varepsilon(0, t) \leq \exp\left(\frac{-m+3r}{\varepsilon}\right) + \frac{2}{\varepsilon} \exp\left(\frac{-r}{\varepsilon}\right) < \delta \quad \text{for all } t \in [0, \exp((m-\delta)/\varepsilon)]. \quad \square$$

We continue with

Proof of Proposition 13. We fix $r \in (0, r_0)$ small enough so that, as in the previous proof, $V_\alpha \leq r$ in B_r , and $m - 5r > M_\alpha$. In view of Lemmata 3 and 4, we may choose $v \in C^2(\overline{\Omega})$, $w \in \text{Lip}(\overline{\Omega})$ and $\eta > 0$ so that, in addition to (4.1), $0 < \min_{\overline{\Omega}} w < \max_{\overline{\Omega}} w < m - 5r$, and, in the viscosity supersolution sense,

$$H_\alpha(x, -Dw) \geq \eta \quad \text{and} \quad D^2w \leq \eta^{-1}I \quad \text{in } \Omega.$$

Setting $u = -w$, $\rho^- = \min_{\overline{\Omega}} w$ and $\rho^+ = \max_{\overline{\Omega}} w$, we get that $\rho^+ < m - 5r$, $0 > -\rho^- \geq u \geq -\rho^+$ on $\overline{\Omega}$ and, in the viscosity subsolution sense,

$$H_\alpha(x, Du) \geq \eta \quad \text{and} \quad D^2u \geq -\eta^{-1}I \quad \text{in } \Omega.$$

For $\varepsilon \in (0, 1)$, we set

$$z^\varepsilon = -\exp\left(\frac{v-m+2r}{\varepsilon}\right) + \exp\left(\frac{u}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right),$$

and find that, in the viscosity subsolution sense,

$$\begin{aligned} \varepsilon \text{tr}[a^\varepsilon D^2 z^\varepsilon] + b \cdot D z^\varepsilon &\geq -\frac{1}{\varepsilon} \exp\left(\frac{v-m+2r}{\varepsilon}\right) (H_\alpha(x, Dv) + \varepsilon \text{tr}[a^\varepsilon D^2 v]) \\ &\quad + \frac{1}{\varepsilon} \exp\left(\frac{u}{\varepsilon}\right) (H_\alpha(x, Du) + \varepsilon \text{tr}[a^\varepsilon D^2 u]) \quad \text{in } Q. \end{aligned}$$

Let $\varepsilon_0 \in (0, 1)$ be a constant to be specified later and assume henceforth that $\varepsilon \in (0, \varepsilon_0)$. Observing that in the viscosity subsolution sense,

$$\varepsilon \text{tr}[a^\varepsilon D^2 u] \geq -\eta^{-1} \text{tr } a^\varepsilon \geq -n(\theta_0 \eta)^{-1} \quad \text{in } Q,$$

and

$$\text{tr}[a^\varepsilon D^2 v] \leq \|D^2 v\|_{L^\infty(\Omega)} \text{tr } a^\varepsilon \leq n\theta_0^{-1} \|D^2 v\|_{L^\infty(\Omega)} \quad \text{in } Q,$$

and, if for $x \in \overline{\Omega}$,

$$\begin{aligned} f(x) &:= -\frac{1}{\varepsilon} \exp\left(\frac{v(x)-m+2r}{\varepsilon}\right) (H_\alpha(x, Dv(x)) + \varepsilon n\theta_0^{-1} \|D^2 v\|_{L^\infty(\Omega)}) \\ &\quad + \frac{1}{\varepsilon} \exp\left(\frac{u(x)}{\varepsilon}\right) (\eta - \varepsilon n(\eta\theta_0)^{-1}), \end{aligned}$$

we obtain, in the viscosity subsolution sense,

$$(4.8) \quad \varepsilon \text{tr}[a^\varepsilon D^2 z^\varepsilon] + b \cdot D z^\varepsilon \geq f \quad \text{in } Q.$$

Choosing $\varepsilon_0 \in (0, 1)$ so that

$$\varepsilon_0 n\theta_0^{-1} \|D^2 v\|_{L^\infty(\Omega)} \leq \min\{\eta, 1\} \quad \text{and} \quad \varepsilon_0 n(\eta\theta_0)^{-1} \leq \frac{\eta}{2},$$

we get

$$\eta - \varepsilon n(\eta\theta_0)^{-1} \geq \frac{\eta}{2} \quad \text{and} \quad H_\alpha(x, Dv) + \varepsilon n\theta_0^{-1} \|D^2 v\|_{L^\infty(\Omega)} \leq \begin{cases} 0 & \text{for all } x \in \Omega \setminus B_r, \\ 2 & \text{for all } x \in B_r, \end{cases}$$

and, accordingly,

$$f \geq \begin{cases} 0 & \text{in } \Omega \setminus B_r, \\ -\frac{2}{\varepsilon} \exp\left(\frac{-m+4r}{\varepsilon}\right) + \frac{\eta}{2\varepsilon} \exp\left(\frac{-\rho^+}{\varepsilon}\right) & \text{in } B_r. \end{cases}$$

Since $\rho^+ < m - 5r$, we have

$$\begin{aligned} -2 \exp\left(\frac{-m+4r}{\varepsilon}\right) + \frac{\eta}{2} \exp\left(\frac{-\rho^+}{\varepsilon}\right) &\geq -2 \exp\left(\frac{-\rho^+ - r}{\varepsilon}\right) + \frac{\eta}{2} \exp\left(\frac{-\rho^+}{\varepsilon}\right) \\ &= \exp\left(\frac{-\rho^+}{\varepsilon}\right) \left(-2 \exp\left(\frac{-r}{\varepsilon}\right) + \frac{\eta}{2}\right). \end{aligned}$$

We may assume by replacing $\varepsilon_0 \in (0, 1)$ by a smaller number that

$$2 \exp\left(\frac{-r}{\varepsilon_0}\right) \leq \frac{\eta}{2},$$

and, therefore,

$$-2 \exp\left(\frac{-m+4r}{\varepsilon}\right) + \frac{\eta}{2} \exp\left(\frac{-\rho^+}{\varepsilon}\right) \geq 0,$$

which ensures that $f \geq 0$ in Ω , and, hence, z^ε , as a function of $(x, t) \in Q$, is a subsolution of (2.1).

Next observe that

$$z^\varepsilon < 0 \quad \text{on } \overline{\Omega},$$

and, if $V_\alpha(x) > m$,

$$z^\varepsilon(x) \leq -\exp\left(\frac{V_\alpha(x) - m + r}{\varepsilon}\right) \leq -\exp\left(\frac{r}{\varepsilon}\right).$$

Fix $C_1 > 0$ so that, for $\varepsilon \in (0, 1)$, $u^\varepsilon \geq -C_1$ on \overline{Q} , and, assume henceforth that $\varepsilon_0 \in (0, 1)$ is small enough so that

$$\exp\left(\frac{r}{\varepsilon_0}\right) \geq C_1.$$

Consequently, we have

$$z^\varepsilon \leq \begin{cases} -\exp\left(\frac{r}{\varepsilon}\right) \leq -C_1 \leq u^\varepsilon & \text{in } (\overline{\Omega} \setminus \Sigma_\alpha^m) \times [0, T(\varepsilon)), \\ 0 \leq u^\varepsilon(\cdot, 0) & \text{in } \Sigma_\alpha^m, \\ 0 \leq u^\varepsilon & \text{in } (\Sigma_\alpha^m \cap \partial\Omega) \times (0, T(\varepsilon)), \end{cases}$$

that is

$$z^\varepsilon \leq u^\varepsilon \quad \text{on } \partial_p Q_{T(\varepsilon)},$$

and, hence, by the comparison principle,

$$z^\varepsilon \leq u^\varepsilon \quad \text{on } \overline{Q}_{T(\varepsilon)}.$$

Finally, we note that

$$\begin{aligned} z^\varepsilon(0) &= -\exp\left(\frac{v(0) - m + 2r}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right) \\ &\geq -\exp\left(\frac{-m+4r}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

which completes the proof. \square

The following corollary is a variant of [11, Theorem 4.1]. Since its proof is similar to the one of Proposition 13 above, here we present only an outline.

Corollary 14. *Assume (1.10), fix $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $T(\varepsilon) \in (0, \infty]$, $C > 0$ and $m > M_\alpha$. If, for $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}; \mathbb{S}^n(\theta_0))$ such that $a^\varepsilon \geq \alpha$ in $Q_{T(\varepsilon)}$, $u^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}) \cap C^{2,1}(Q_{T(\varepsilon)})$ is a supersolution of (2.1) in $Q_{T(\varepsilon)}$ such that*

$$u^\varepsilon \geq 0 \quad \text{in } (\Sigma_\alpha^m \cap \partial\Omega) \times (0, T(\varepsilon)) \quad \text{and } u^\varepsilon \geq -C \quad \text{in } Q_{T(\varepsilon)},$$

then, for any $\delta > 0$, there exists $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$u^\varepsilon(0, t) \geq -\delta \quad \text{for all } t \in [\exp(m/\varepsilon), T(\varepsilon)).$$

Outline of proof. Let $r \in (0, r_0)$, $\eta > 0$, ρ^\pm , $v \in C^2(\overline{\Omega})$, and $w, u, z^\varepsilon, f \in \text{Lip}(\overline{\Omega})$ be the same as in the proof of Proposition 13. According to (4.8), we have in the viscosity subsolution sense,

$$\varepsilon \text{tr}[a^\varepsilon D^2 z^\varepsilon] + b \cdot D z^\varepsilon \geq f \quad \text{in } Q.$$

Choosing $\varepsilon_0 \in (0, 1)$ so that

$$\varepsilon_0 n \theta_0^{-1} \|D^2 v\|_{L^\infty(\Omega)} \leq \min\{\eta, 1\}, \quad \varepsilon_0 n (\eta \theta)^{-1} \leq \frac{\eta}{2} \quad \text{and} \quad 2 \exp\left(\frac{-r}{\varepsilon_0}\right) \leq \frac{\eta}{4}$$

and noting that, in $\Omega \setminus B_r$ and $\varepsilon \in (0, \varepsilon_0)$,

$$f \geq \frac{1}{\varepsilon} \exp\left(\frac{u}{\varepsilon}\right) (\eta - \varepsilon n (\eta \theta)^{-1}) \geq \frac{\eta}{2} \exp\left(\frac{-\rho^+}{\varepsilon}\right),$$

we compute, as in the proof of Proposition 13, to get, for any $\varepsilon \in (0, \varepsilon_0)$, that

$$f \geq \exp\left(\frac{-\rho^+}{\varepsilon}\right) \left(-2 \exp\left(\frac{-r}{\varepsilon}\right) + \frac{\eta}{2}\right) \geq \frac{\eta}{4} \exp\left(\frac{-\rho^+}{\varepsilon}\right) \quad \text{in } \Omega.$$

Now, we fix $\gamma \in (0, \eta]$, set, for $(x, t) \in \overline{Q}$ and $\varepsilon \in (0, \varepsilon_0)$,

$$g^\varepsilon(x, t) := z^\varepsilon(x) - C + \frac{\gamma t}{4} \exp\left(\frac{-\rho^+}{\varepsilon}\right),$$

and observe that, for each $\varepsilon \in (0, \varepsilon_0)$, g^ε is a subsolution of (2.1).

Let

$$\tau(\varepsilon) = \frac{4C}{\gamma} \exp\left(\frac{\rho^+}{\varepsilon}\right),$$

and observe that, for any $(x, t) \in \overline{\Omega} \times [0, \tau(\varepsilon)]$ such that $V_\alpha(x) > m$,

$$g^\varepsilon(x, t) \leq z^\varepsilon(x) \leq -\exp\left(\frac{r}{\varepsilon}\right).$$

We may assume by replacing $\varepsilon_0 > 0$ by a smaller number if necessary that

$$\exp\left(\frac{r}{\varepsilon_0}\right) \geq C.$$

Accordingly, we have

$$g^\varepsilon \leq \begin{cases} -\exp\left(\frac{r}{\varepsilon}\right) \leq -C \leq u^\varepsilon & \text{in } (\overline{\Omega} \setminus \Sigma_\alpha^m) \times [0, T(\varepsilon) \wedge \tau(\varepsilon)), \\ -C \leq u^\varepsilon(\cdot, 0) & \text{in } \Sigma_\alpha^m \times \{0\}, \\ 0 \leq u^\varepsilon & \text{in } (\Sigma_\alpha^m \cap \partial\Omega) \times (0, T(\varepsilon) \wedge \tau(\varepsilon)). \end{cases}$$

Thus,

$$g^\varepsilon \leq u^\varepsilon \quad \text{in } \partial_p Q_{T(\varepsilon) \wedge \tau(\varepsilon)},$$

and, hence, by the comparison principle

$$g^\varepsilon \leq u^\varepsilon \quad \text{in } \overline{Q}_{T(\varepsilon) \wedge \tau(\varepsilon)}.$$

The final step begins by noting that

$$u^\varepsilon(0, \tau(\varepsilon)) \geq g^\varepsilon(0, \tau(\varepsilon)) = z^\varepsilon(0)$$

and

$$z^\varepsilon(0) \geq -\exp\left(\frac{-m+4r}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Fix $\delta > 0$ and, if necessary, replace ε_0 by a smaller number such that $z^\varepsilon(0) \geq -\delta$ for all $\varepsilon \in (0, \varepsilon_0)$. Recalling the definition of $\tau(\varepsilon)$ and observing that

$$u^\varepsilon(0, t) \geq z^\varepsilon(0) \quad \text{if } t = \frac{4C}{\gamma} \exp\left(\frac{\rho^+}{\varepsilon}\right) < T(\varepsilon) \quad \text{and } 0 < \gamma \leq \eta,$$

we conclude that

$$u^\varepsilon(0, t) \geq -\delta \quad \text{for all } t \in [(4C/\eta) \exp(\rho^+/\varepsilon), T(\varepsilon)) \quad \text{and } \varepsilon \in (0, \varepsilon_0).$$

Since $\rho^+ < m - 5r$, by selecting $\varepsilon_0 \in (0, 1)$ sufficiently small, we may assume that $(4C/\eta) \exp(\rho^+/\varepsilon) \leq \exp(m/\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$, which completes the proof. \square

5. THE PROOFS OF PROPOSITIONS 2, 3 AND 4

Proof of Proposition 2. Since the arguments are similar for both cases when $\beta_1 < \beta_2$ and $\beta_1 > \beta_2$, here we treat only the case $\beta_1 < \beta_2$.

We argue by contradiction and suppose that

$$(5.1) \quad \limsup_{k \rightarrow \infty} \lambda_k < M(\beta_2).$$

Let $\delta > 0$ be a constant to be fixed later, define α_δ^+ and \mathcal{H}_δ^+ as in Section 3, with β_0 replaced by β_2 , and, as in Section 3, let V_δ^+ be the maximal subsolution of

$$\mathcal{H}_\delta^+(x, Du) = 0 \quad \text{in } \Omega, \quad u(0) = 0,$$

and set $M_\delta^+ = \min_{\partial\Omega} V_\delta^+$.

Since Proposition 11 yields

$$\lim_{\delta \rightarrow 0^+} M_\delta^+ = M(\beta_2),$$

in view of (5.1), we may choose $\delta > 0$ so that

$$\limsup_{k \rightarrow \infty} \lambda_k + \delta < M_\delta^+.$$

We fix $m \in \mathbb{R}$ so that

$$\limsup_{k \rightarrow \infty} \lambda_k + \delta < m < M_\delta^+,$$

and, by passing to a subsequence if necessary, we may assume that

$$\lambda_k \leq m - \delta \quad \text{for all } k \in \mathbb{N}.$$

Set

$$\Sigma = \{x \in \overline{\Omega} : V_\delta^+(x) \leq m\},$$

and note that Σ is a compact subset of Ω .

In view of the continuity of the map $t \mapsto u^\varepsilon(0, t)$, reselecting, if needed, β_1 , μ_k and λ_k , we may assume that, for all $t \in [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)]$ and $k \in \mathbb{N}$,

$$(5.2) \quad \beta_2 - \frac{\delta}{2} < \beta_1 \leq u^{\varepsilon_k}(0, t) \leq \beta_2.$$

Now we choose $\gamma \in (0, \delta/2)$ small enough, so that

$$(5.3) \quad \Sigma \subset \Omega_\gamma \quad \text{and} \quad \beta_2 - \beta_1 > 2\gamma.$$

Proposition 10 gives $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$,

$$(5.4) \quad |u^\varepsilon(x, t) - u^\varepsilon(0, t)| < \gamma \quad \text{for all } (x, t) \in \Omega_\gamma \times [\exp(a_1/\varepsilon), \infty).$$

We assume that $\varepsilon_k < \varepsilon_0$ for all $k \in \mathbb{N}$, and combine (5.4) and (5.2), to find

$$(5.5) \quad |u^{\varepsilon_k}(x, t) - \beta_2| \leq \delta \quad \text{for all } (x, t) \in \Omega_\gamma \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)],$$

and

$$u^{\varepsilon_k}(x, \exp(\mu_k/\varepsilon_k)) \leq \beta_1 + \gamma \quad \text{for all } x \in \Omega_\gamma \text{ and } k \in \mathbb{N}.$$

Since (5.5) implies that

$$a(x, u^{\varepsilon_k}(x, t)) \leq \alpha_\delta(x) \quad \text{for all } (x, t) \in \Omega \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)], \quad k \in \mathbb{N},$$

setting

$$\begin{cases} v^k(x, t) = u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 - \gamma, \\ a^k(x, t) = a(x, u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k))), \end{cases}$$

we see that

$$v_t^k = \varepsilon_k \operatorname{tr}[a^k(x, t) D^2 v^k] + b(x) \cdot D v^k \quad \text{for all } (x, t) \in Q.$$

Furthermore, since $v^k(\cdot, 0) \leq 0$ in Ω_γ , it follows that

$$v^k(\cdot, 0) \leq 0 \quad \text{in } \Sigma.$$

An application of Proposition 12, with ε_k , v^k and γ in place of ε , u^ε and δ respectively, guarantees that, for sufficiently large k , we have

$$v^k(0, t) \leq \gamma \quad \text{for all } t \in [0, \exp(\lambda_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)],$$

which, in particular, yields

$$v^k(0, \exp(\lambda_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)) \leq \gamma.$$

This shows that

$$u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) \leq \beta_1 + 2\gamma < \beta_2,$$

which is a contradiction. \square

Proof of Proposition 3. Since the arguments are similar, here we only consider the case where $\beta_2 < \beta_1$ holds.

We suppose that

$$(5.6) \quad G^-(\beta_2) > \beta_2,$$

and obtain a contradiction.

For a small constant $\delta > 0$ to be chosen later, define α_δ^- and \mathcal{H}_δ^- as in Section 3, with β_0 replaced by β_2 , let V_δ^- be the quasi-potential corresponding to (α_δ^-, b) , that is the maximal subsolution of

$$\mathcal{H}_\delta^-(x, Du) = 0 \quad \text{in } \Omega \quad \text{and} \quad u(0) = 0.$$

and V^{β_2} the quasi-potential corresponding to the pair $(a(\cdot, \beta_2), b)$, set

$$M_\delta^- = \min_{\partial\Omega} V_\delta^-, \quad \Gamma_\delta^- = \arg \min(V_\delta^- | \partial\Omega) \quad \text{and} \quad \Gamma^{\beta_2} = \arg \min(V^{\beta_2} | \partial\Omega),$$

and observe that

$$G^-(\beta_2) = \min_{\Gamma^{\beta_2}} g.$$

Due to (5.6), we have

$$\min_{\Gamma^{\beta_2}} g > \beta_2.$$

Furthermore, in view of (3.5), we may choose $\delta > 0$ so that

$$(5.7) \quad \min_{\Gamma_\delta^-} g > \beta_2 + \delta.$$

Finally replacing, if necessary, β_1 , μ_k and λ_k we may assume that

$$\beta_1 \geq u^\varepsilon(0, t) \geq \beta_2 \quad \text{for all } t \in [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)], \quad k \in \mathbb{N},$$

and

$$(5.8) \quad \beta_1 < \beta_2 + \delta/2.$$

Since, by the maximum principle, $g_{\min} \leq u^\varepsilon \leq g_{\max}$ in \overline{Q} , we find that Theorem 10 yields $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$(5.9) \quad |u^\varepsilon - u^\varepsilon(0, t)| < \delta/2 \quad \text{in } \Omega_{\delta/2} \times [\exp(a_1/\varepsilon), \infty).$$

Consequently, if $k \in \mathbb{N}$ is sufficiently large, then $\varepsilon_k < \varepsilon_0$ and

$$(5.10) \quad |u^{\varepsilon_k} - \beta_2| < \delta \quad \text{in } \Omega_{\delta/2} \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)].$$

Henceforth, passing if necessary to a subsequence, we assume that (5.10) holds for all $k \in \mathbb{N}$ and, thus

$$(5.11) \quad \alpha_\delta^-(x) \leq a(x, u^{\varepsilon_k}(x, t)) \quad \text{for all } (x, t) \in \overline{\Omega} \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)], \quad k \in \mathbb{N}.$$

We set $\Pi = \{x \in \partial\Omega : g(x) > \beta_2 + \delta\}$ and note that, in view of (5.7), Π is an open neighborhood, relative to $\partial\Omega$, of Γ_δ^- and

$$\{x \in \overline{\Omega} : V_\delta^-(x) \leq M_\delta^-\} = \{x \in \Omega : V_\delta^-(x) \leq M_\delta^-\} \cup \Gamma_\delta^- \subset \Omega^\Pi,$$

and deduce, for $\gamma > 0$ sufficiently small,

$$(5.12) \quad \{x \in \overline{\Omega} : V_\delta^-(x) \leq M_\delta^- + \gamma\} \subset \Omega_\gamma^\Pi.$$

In view of (5.8), we observe that

$$(5.13) \quad g > \beta_2 + \delta > \beta_1 \quad \text{in } \Pi.$$

We fix $\gamma > 0$ so that (5.12) and $5\gamma < \beta_1 - \beta_2$ hold, set

$$\Sigma := \{x \in \overline{\Omega} : V_\delta^-(x) \leq M_\delta^- + \gamma\} \subset \Omega_\gamma^\Pi.$$

Noting that $\beta_1 > \beta_1 - 4\gamma > \beta_2$, we select a sequence $\{\nu_k\}$ so that

$$(5.14) \quad \begin{cases} \mu_k < \nu_k < \lambda_k, & u^{\varepsilon_k}(0, \exp(\nu_k/\varepsilon_k)) = \beta_1 - 3\gamma \quad \text{for all } k \in \mathbb{N}, \\ \beta_1 \geq u^{\varepsilon_k}(0, t) \geq \beta_1 - 3\gamma & \text{for all } t \in [\exp(\mu_k/\varepsilon_k), \exp(\nu_k/\varepsilon_k)], \quad k \in \mathbb{N}. \end{cases}$$

Now we show that, for some $\rho > 0$ and sufficiently large $k \in \mathbb{N}$,

$$(5.15) \quad \exp(\nu_k/\varepsilon_k) \geq \exp(\mu_k/\varepsilon_k) + \exp(\rho/\varepsilon_k).$$

For this, similarly to (5.9), we use Proposition 10, to find that, for some $r \in (0, r_0)$ and sufficiently large $k \in \mathbb{N}$,

$$|u^{\varepsilon_k} - u^{\varepsilon_k}(0, \cdot)| < \gamma \quad \text{in } B_r \times [\exp(a_1/\varepsilon_k), \infty).$$

For every such large $k \in \mathbb{N}$, we set

$$v^k(x, t) := u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 + \gamma \quad \text{for } (x, t) \in \overline{Q},$$

and note that

$$v^k(\cdot, 0) \geq 0 \quad \text{in } B_r.$$

We apply Proposition 12, with ε , u^ε and α replaced respectively by ε_k , $-v^k$ and $\theta_0^{-1}I$, to deduce that, for some $\rho > 0$,

$$-v^k(0, t) \leq \gamma \quad \text{for all } t \in [0, \exp(\rho/\varepsilon_k)],$$

that is,

$$u^{\varepsilon_k}(0, t) \geq \beta_1 - 2\gamma \quad \text{for all } t \in [\exp(\mu_k/\varepsilon_k), \exp(\mu_k/\varepsilon_k) + \exp(\rho/\varepsilon_k)],$$

which, in view of the choice of ν_k , implies that (5.15) holds for sufficiently large $k \in \mathbb{N}$.

In what follows we may assume by replacing if necessary $\{\varepsilon_k\}$ by a further subsequence that (5.15) is satisfied for some $\rho > 0$ and all $k \in \mathbb{N}$. We set

$$w^k(x, t) = u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 + 3\gamma \quad \text{for } (x, t) \in \overline{Q}, \quad k \in \mathbb{N},$$

and note that, in view of (5.14) and (5.13),

$$\begin{cases} w^k(0, t) \geq 0 & \text{for all } t \in [0, \exp(\nu_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)] \\ w^k(x, t) = g(x) - \beta_1 + 3\gamma \geq 0 & \text{for all } (x, t) \in \Pi \times [0, \infty). \end{cases}$$

Recalling (5.15), we apply Theorem 9, with ε and u^ε replaced by ε_k and $-w_k$, to get, for sufficiently large k ,

$$-w^k(x, \exp(\nu_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)) \leq \gamma \quad \text{for all } x \in \Omega_\gamma^\Pi,$$

which reads

$$u^{\varepsilon_k}(x, \exp(\nu_k/\varepsilon_k)) \geq \beta_1 - 4\gamma \quad \text{for all } x \in \Omega_\gamma^\Pi.$$

Finally, for $(x, t) \in \overline{Q}$, we set

$$z^k(x, t) = u^{\varepsilon_k}(x, t + \exp(\nu_k/\varepsilon_k)) - \beta_1 + 4\gamma,$$

observe that, if $k \in \mathbb{N}$ is sufficiently large, then

$$z^k(\cdot, 0) \geq 0 \quad \text{in } \Sigma \quad \text{and} \quad z^k = g - \beta_1 + 4\gamma \geq 0 \quad \text{in } \Pi \times [0, \infty),$$

and invoke Proposition 13, to conclude that, for sufficiently large $k \in \mathbb{N}$,

$$z^k(0, \exp(\lambda_k/\varepsilon_k) - \exp(\nu_k/\varepsilon_k)) \geq -\gamma,$$

and, hence,

$$u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) \geq \beta_1 - 5\gamma > \beta_2,$$

which is a contradiction. \square

Proof of Proposition 4. Since the arguments are similar, we give the proof under the assumption that

$$(5.16) \quad G^-(\beta_0) > \beta_0.$$

We suppose that

$$(5.17) \quad \rho_0 > M(\beta_0),$$

and obtain a contradiction.

Fix $\delta > 0$ and let α_δ^- and \mathcal{H}_δ^- as in Section 3 and V_δ^- and V^{β_0} be the quasi-potentials corresponding to (α_δ^-, b) and $(a(\cdot, \beta_0), b)$ respectively, set

$$M_\delta^- = \min_{\partial\Omega} V_\delta^-, \quad \Gamma_\delta^- = \arg \min(V_\delta^- | \partial\Omega) \quad \text{and} \quad \Gamma^{\beta_0} = \arg \min(V^{\beta_0} | \partial\Omega),$$

and note, in view of Proposition 11, (5.16) and (5.17), that

$$\liminf_{\delta \rightarrow 0+} \min_{\Gamma_\delta^-} g \geq \min_{\Gamma^{\beta_0}} g = G^-(\beta_0) > \beta_0 \quad \text{and} \quad \lim_{\delta \rightarrow 0+} M_\delta^- = M(\beta_0) < \rho_0.$$

Choose $\delta > 0$ so that

$$(5.18) \quad \min_{\Gamma_\delta^-} g > 4\delta + \beta_0.$$

For $m > M_\delta^-$ set

$$\Sigma^m := \{x \in \overline{\Omega} : V_\delta^-(x) \leq m\}$$

and note that

$$\limsup_{m \rightarrow M_\delta^- + 0} \Sigma^m \cap \partial\Omega = \{x \in \overline{\Omega} : V_\delta^-(x) \leq M_\delta^-\} \cap \partial\Omega = \Gamma_\delta^-.$$

Hence, we may choose $m \in (M_\delta^-, \rho_0]$ so that

$$(5.19) \quad \Sigma^m \cap \partial\Omega \subset \{x \in \partial\Omega : g(x) > \beta_0 + 3\delta\}.$$

The maximum principle yields that, for all $(x, t) \in Q$ and $\varepsilon \in (0, 1)$, $u^\varepsilon(x, t) \in I_{g\cdot}$, while Theorem 10 implies the existence of $\varepsilon_0 \in (0, 1)$ such that for all $(x, t) \in \Omega_{\delta/2} \times [\exp((\rho_0 - \delta)/\varepsilon), \infty)$ and $\varepsilon \in (0, \varepsilon_0)$,

$$|u^\varepsilon(x, t) - u^\varepsilon(0, t)| < \frac{\delta}{2}.$$

Our assumptions yield $\gamma > 0$ and a sequence $\{\varepsilon_k\} \subset (0, \varepsilon_0)$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and, for all $\rho \in [\rho_0 - \gamma, \rho_0 + \gamma] \subset (0, \infty)$ and $k \in \mathbb{N}$,

$$u^{\varepsilon_k}(0, \exp(\rho/\varepsilon_k)) \in [\beta_0 - \delta/2, \beta_0 + \delta/2].$$

Hence, if $(x, \rho) \in \Omega_{\delta/2} \times [\rho_0 - \gamma, \rho_0 + \gamma]$ and $k \in \mathbb{N}$, we get

$$(5.20) \quad u^{\varepsilon_k}(x, \exp(\rho/\varepsilon_k)) \in (\beta_0 - \delta, \beta_0 + \delta),$$

and, moreover,

$$(5.21) \quad \alpha_\delta^-(x) \leq a(x, u^{\varepsilon_k}(x, \exp(\rho/\varepsilon_k))).$$

Set

$$v^\varepsilon(x, t) := u^\varepsilon(x, t + \exp((\rho_0 - \gamma)/\varepsilon)) - \beta_0 - 3\delta \quad \text{for all } (x, t) \in Q, \quad \varepsilon \in (0, 1),$$

and

$$a^\varepsilon(x, t) := a(x, u^\varepsilon(x, t + \exp((\rho_0 - \gamma)/\varepsilon))) \quad \text{for all } (x, t) \in Q, \quad \varepsilon \in (0, 1),$$

and observe that, for $T_k := \exp((\rho_0 + \gamma)/\varepsilon_k) - \exp((\rho_0 - \gamma)/\varepsilon_k)$, v^ε is a solution of (2.1) and (1.2),

$$\alpha_\delta^-(x) \leq a^{\varepsilon_k}(x, t) \quad \text{for all } (x, t) \in \Omega \times [0, T_k], \quad k \in \mathbb{N},$$

and

$$v^{\varepsilon_k}(x, t) = g(x) - \beta_0 - 3\delta > 0 \quad \text{for all } (x, t) \in \Sigma^m \cap \partial\Omega \times [0, T_k]$$

In view of Corollary 14, we may assume, by passing to a subsequence, that

$$v^{\varepsilon_k}(0, t) \geq -\delta \quad \text{for all } t \in [\exp(m/\varepsilon_k), T_k] \quad \text{and } k \in \mathbb{N}.$$

Since $T_k > \exp(m/\varepsilon_k)$ for sufficiently large $k \in \mathbb{N}$, we find $k \in \mathbb{N}$ such that

$$v^{\varepsilon_k}(0, T_k) \geq -\delta,$$

which yields the contradiction

$$u^{\varepsilon_k}(0, \exp((\rho_0 + \gamma)/\varepsilon_k)) \geq \beta_0 + 3\delta.$$

□

6. THE PROOF OF THE MAIN THEOREM

The proof of Theorem 1 is a relatively easy consequence of Propositions 2, 3 and 4 as shown in [6, 8]. For the reader's convenience, we reproduce it here. We begin with two lemmata.

Lemma 5. *Assume (1.10) and let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (1.1) and (1.2). For any $\delta > 0$ there exist $\lambda_0 > 0$ and $\varepsilon_0 \in (0, 1)$ such that*

$$(6.1) \quad |u^\varepsilon(0, t) - g(0)| \leq \delta \quad \text{for all } t \in [0, \exp(\lambda_0/\varepsilon)] \quad \text{and } \varepsilon \in (0, \varepsilon_0).$$

Proof. Let $V \in \text{Lip}(\overline{\Omega})$ be the quasi-potential associated with $(\theta_0^{-1}I, b)$. We choose $m > 0$ small enough so that $m < \min_{\partial\Omega} V$ and

$$\{x \in \Omega : V(x) \leq m\} \subset \{x \in \Omega : |g(x) - g(0)| \leq \delta/2\}.$$

Applying Proposition 12, with $a^\varepsilon(x, t) = a(x, u^\varepsilon(x, t))$ and $\alpha(x) = \theta_0^{-1}I$ and u^ε replaced by $\pm(u^\varepsilon - g(0)) - \delta/2$, we get that, for each $\gamma > 0$, there is $\varepsilon_0 \in (0, 1)$ such that

$$\pm(u^\varepsilon(0, t) - g(0)) - \delta/2 \leq \gamma \quad \text{for all } t \in [0, \exp((m - \gamma)/\varepsilon)] \quad \text{and } \varepsilon \in (0, \varepsilon_0).$$

We fix $\gamma > 0$ small enough so that $\gamma < \min\{\delta/2, m\}$, and we get (6.1) with $\lambda_0 = m - \gamma$. □

Lemma 6. *Assume (1.10) and (1.17) and let $\lambda > 0$ and, for each $\varepsilon \in (0, 1)$, $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of (1.1) and (1.2). If $c_1 > c_0$, then*

$$(6.2) \quad \liminf_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \geq \bar{c}(\lambda),$$

and, if $c_1 < c_0$, then

$$(6.3) \quad \limsup_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq \bar{c}(\lambda).$$

This lemma is exactly the same as [8, Lemma 3.12].

Proof. We give only the proof of the first assertion, since the other claim can be proved similarly.

Note that, in view of the definition of c_1 and the function \bar{c} , $G^-(c) > c$ for all $c \in [c_0, c_1]$ and $\lambda \neq M(c)$ for all $c \in [c_0, \bar{c}(\lambda))$. Furthermore, since the function M is continuous, we have $\lambda > M(c)$ for all $c \in [c_0, \bar{c}(\lambda))$.

We show first that, for any $\rho > 0$,

$$(6.4) \quad \liminf_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\rho/\varepsilon)) \geq c_0.$$

According to Lemma 5, there exists $\lambda_0 > 0$ such that, for any $\rho \in (0, \lambda_0]$,

$$(6.5) \quad \lim_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\rho/\varepsilon)) = c_0,$$

which shows that (6.4) holds if $\rho \leq \lambda_0$.

Fix any $\rho > \lambda_0$ and, to prove (6.4), suppose to the contrary that

$$\liminf_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\rho/\varepsilon)) < c_0.$$

It is easily seen that there exist sequences $\{\varepsilon_k\}$, $\{\mu_k\}$ and $\{\lambda_k\}$ of positive numbers and two constants $\beta_1, \beta_2 \in I_g$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, $c_0 > \beta_1 > \beta_2$, and, for all $k \in \mathbb{N}$,

$$\lambda_0 < \mu_k < \lambda_k \leq \rho, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

Since $G^-(c_0) > c_0$ and G^- is lower semicontinuous, we may assume reselecting β_1, β_2 close enough to c_0 so that $G^-(\beta_2) > \beta_2$. This contradicts Proposition 3, which proves that (6.4) holds.

To show (6.2), in view of (6.4), we may assume that $\bar{c}(\lambda) > c_0$ and suppose that (6.2) is false, that is,

$$(6.6) \quad \liminf_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\lambda/\varepsilon)) < \bar{c}(\lambda),$$

which in turn implies together with (6.4) that $\bar{c}(\lambda) > c_0$.

We set

$$\hat{c}(\rho) := \liminf_{\varepsilon \rightarrow 0+} u^\varepsilon(0, \exp(\rho/\varepsilon)) \quad \text{for } \rho \in (0, \lambda],$$

and show, arguing by contradiction, that

$$(6.7) \quad \hat{c}(\rho_1) \leq \hat{c}(\rho_2) \quad \text{if } 0 < \rho_1 < \rho_2 \leq \lambda.$$

To this end, we suppose to the contrary that there exist $0 < \rho_1 < \rho_2 \leq \lambda$ such that

$$\hat{c}(\rho_1) > \hat{c}(\rho_2).$$

We may assume that $\hat{c}(\rho_2) \leq \hat{c}(\lambda)$. Indeed, if $\hat{c}(\rho_2) > \hat{c}(\lambda)$, then, replacing ρ_2 by λ , we find $\hat{c}(\rho_1) > \hat{c}(\rho_2)$ and $\hat{c}(\rho_2) = \hat{c}(\lambda)$. Now, noting by (6.6) and (6.5) that $c_0 \leq \hat{c}(\rho_2) \leq \hat{c}(\lambda) < \bar{c}(\lambda)$, we may choose sequences $\{\varepsilon_k\}$, $\{\mu_k\}$, $\{\lambda_k\}$ of positive numbers and constants $\beta_1, \beta_2 \in I_g$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, $\bar{c}(\lambda) > \beta_1 > \beta_2 > c_0$, and, for all $k \in \mathbb{N}$,

$$u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1, \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2 \quad \text{and} \quad \rho_1 < \mu_k < \lambda_k \leq \rho_2.$$

Here and there, for notational simplicity, we use the same symbols β_i , ε_k , μ_k and λ_k to denote different quantities in different arguments. Moreover, since $c_0 < \beta_2 < \bar{c}(\lambda)$, we have $G^-(\beta_2) > \beta_2$. Thus, we are in the situation that contradicts Proposition 3, and we conclude that (6.7) holds.

The last step of our proof is an application of Proposition 4 for a contradiction.

In view of the monotonicity (6.7), the function \hat{c} has at most countably many discontinuities on $(0, \lambda]$ and, recalling that $c_0 \leq \hat{c}(\rho) < \bar{c}(\lambda)$ for all $\rho \in (0, \lambda]$ and that $G^-(c) > c$ and $M(c) < \lambda$ for all $c \in [c_0, \bar{c}(\lambda))$, we may choose $\rho_0 \in (0, \lambda)$ so that \hat{c} is continuous at ρ_0 and, for $\beta_0 := \hat{c}(\rho_0)$,

$$(6.8) \quad \rho_0 > M(\beta_0).$$

We fix any $\delta > 0$. Since $G^-(\beta_0) > \beta_0$, in view of the lower semicontinuity of G^- , we may choose $\delta_1 \in (0, \delta/3)$ so that

$$(6.9) \quad G^-(c) > \beta_0 + 3\delta_1 \quad \text{for all } c \in [\beta_0 - 3\delta_1, \beta_0 + 3\delta_1].$$

Moreover the continuity of \hat{c} at ρ_0 yields $\gamma > 0$ such that

$$[\rho_0 - \gamma, \rho_0 + \gamma] \subset (0, \lambda),$$

and, for all $\rho \in [\rho_0 - \gamma, \rho_0 + \gamma]$,

$$(6.10) \quad \hat{c}(\rho) \in [\beta_0 - \delta_1, \beta_0 + \delta_1].$$

Now, we show that there exists a sequence $\{\varepsilon_k\} \subset (0, 1)$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and, for all $\rho \in [\rho_0 - \gamma, \rho_0 + \gamma]$ and $k \in \mathbb{N}$,

$$(6.11) \quad u^{\varepsilon_k}(0, \exp(\rho/\varepsilon_k)) \in [\beta_0 - 3\delta_1, \beta_0 + 3\delta_1].$$

Indeed, in view of the definition of \hat{c} and (6.10), we may choose a sequence $\{\varepsilon_k\} \subset (0, 1)$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and, for all $k \in \mathbb{N}$,

$$(6.12) \quad u^{\varepsilon_k}(0, \exp((\rho_0 + \gamma)/\varepsilon_k)) \in [\beta_0 - 2\delta_1, \beta_0 + 2\delta_1].$$

Since $\hat{c}(\rho_0 - \gamma) \geq \beta_0 - \delta_1$, (6.10) gives

$$\liminf_{k \rightarrow \infty} u^{\varepsilon_k}(0, \exp((\rho_0 - \gamma)/\varepsilon_k)) \geq \beta_0 - \delta_1,$$

and, therefore, by passing to a subsequence if necessary, we may assume that, for all $k \in \mathbb{N}$,

$$(6.13) \quad u^{\varepsilon_k}(0, \exp((\rho_0 - \gamma)/\varepsilon_k)) \geq \beta_0 - 2\delta_1.$$

To complete the proof of (6.11), we need only to show that for infinitely many $k \in \mathbb{N}$ and all $\rho \in [\rho_0 - \gamma, \rho_0 + \gamma]$,

$$(6.14) \quad u^{\varepsilon_k}(0, \exp(\rho/\varepsilon_k)) \in [\beta_0 - 3\delta_1, \beta_0 + 3\delta_1].$$

If this is not the case, there exist a subsequence of $\{\varepsilon_k\}$, which we denote again by the same symbol, and a sequence $\{\rho_k\} \subset [\rho_0 - \gamma, \rho_0 + \gamma]$ such that either

$$(6.15) \quad u^{\varepsilon_k}(0, \exp(\rho_k/\varepsilon_k)) > \beta_0 + 3\delta_1 \quad \text{for all } k \in \mathbb{N},$$

or

$$(6.16) \quad u^{\varepsilon_k}(0, \exp(\rho_k/\varepsilon_k)) < \beta_0 - 3\delta_1 \quad \text{for all } k \in \mathbb{N},$$

In view of (6.12), if (6.15) holds, then there are two sequences $\{\mu_k\}, \{\lambda_k\} \subset (0, \lambda)$ such that, for all $k \in \mathbb{N}$,

$$(6.17) \quad \begin{cases} \beta_0 + 3\delta_1 = u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) > u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_0 + 2\delta_1, \\ \rho_0 - \gamma \leq \mu_k < \lambda_k \leq \rho_0 + \gamma. \end{cases}$$

Similarly, in view of (6.13), if (6.16) holds, then there are two sequences $\{\mu_k\}, \{\lambda_k\} \subset (0, \lambda)$ such that, for all $k \in \mathbb{N}$,

$$(6.18) \quad \begin{cases} \beta_0 - 2\delta_1 = u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) > u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_0 - 3\delta_1, \\ \rho_0 - \gamma \leq \mu_k < \lambda_k < \rho_0 + \gamma. \end{cases}$$

If (6.17) holds, setting $\beta_1 := \beta_0 + 3\delta_1$ and $\beta_2 := \beta_0 + 2\delta_1$ and noting by (6.9) that $G^-(\beta_2) > \beta_2$, we apply Proposition 3, to obtain a contradiction.

In the case (6.18) holds, setting $\beta_1 := \beta_0 - 2\delta_1$ and $\beta_2 := \beta_0 - 3\delta_1$ and noting that $G^-(\beta_2) > \beta_2$, we get a contradiction by Proposition 3.

Now, we find that (6.14) holds and, therefore, there is a sequence $\{\varepsilon_k\} \subset (0, 1)$ for which (6.11) holds.

Thus, under the supposition (6.6), we have shown that (6.8) and (6.10) hold for some sequence $\{\varepsilon_k\} \subset (0, 1)$ converging to zero. Proposition 4 assures that $\rho_0 \leq M(\beta_0)$, which contradicts (6.8). Therefore, we conclude that (6.2) must hold. \square

Proof of Theorem 1. In view of Theorem 10, we only need to show that

$$(6.19) \quad \lim_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) = \bar{c}(\lambda).$$

The comparison principle yields that

$$g_{\min} \leq u^\varepsilon \leq g_{\max} \text{ on } \overline{Q}.$$

We fix $\lambda > 0$ and consider first the case $\lambda < M(c_0)$, which implies that $\bar{c}(\lambda) = c_0$, and prove that

$$(6.20) \quad \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq \bar{c}(\lambda) = c_0.$$

We argue by contradiction and suppose that

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > c_0.$$

Using the continuity of the function M , we choose $\beta_1, \beta_2 \in \mathbb{R}$ so that

$$(6.21) \quad c_0 < \beta_1 < \beta_2 < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \quad \text{and} \quad M(\beta_2) > \lambda,$$

and note that, in view of Lemma 5, there are constants $\lambda_0 \in (0, \lambda)$ and $\varepsilon_0 \in (0, 1)$ such that

$$(6.22) \quad u^\varepsilon(0, \exp(\lambda_0/\varepsilon)) \leq \beta_1 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

On the other hand, (6.21) yields a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ such that $\varepsilon_k \rightarrow 0$ and

$$u^{\varepsilon_k}(0, \exp(\lambda/\varepsilon_k)) \geq \beta_2 \quad \text{for all } k \in \mathbb{N},$$

while, (6.22) gives

$$u^{\varepsilon_k}(0, \exp(\lambda_0/\varepsilon_k)) \leq \beta_1 \quad \text{for all } k \in \mathbb{N}.$$

The continuity of $t \mapsto u^{\varepsilon_k}(0, t)$ implies that, for each $k \in \mathbb{N}$, there exist $\mu_k, \lambda_k \in [\lambda_0, \lambda]$ such that $\lambda_0 \leq \mu_k < \lambda_k \leq \lambda$ and

$$u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

Proposition 2 now assures that $\limsup_{k \rightarrow \infty} \lambda_k \geq M(\beta_2)$, but this contradicts that $\lambda_k \leq \lambda < M(\beta_2)$ for all $k \in \mathbb{N}$.

A similar argument shows that

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \geq \bar{c}(\lambda),$$

and, thus, we have (6.19) in the case where $\lambda < M(c_0)$.

Next we consider the case where $\lambda \geq M(c_0)$ and $c_1 = c_0$ and recall that, by definition, $\bar{c}(\lambda) = c_0$. We first suppose that

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > c_0,$$

and use (1.15) and the upper semicontinuity of G^+ , to select $\beta_2 \in \mathbb{R}$ so that $c_0 < \beta_2 < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$ and $G^+(\beta_2) < \beta_2$.

Choosing, for instance, $\beta_1 = (c_0 + \beta_2)/2$, so that $c_0 < \beta_1 < \beta_2$, and, using Lemma 5 as in the previous case, we may choose sequences $\{\varepsilon_k\}$, $\{\mu_k\}$, $\{\lambda_k\}$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and for some $\lambda_0 > 0$ and all $k \in \mathbb{N}$,

$$\lambda_0 \leq \mu_k < \lambda_k \leq \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

This contradicts Proposition 3, and thus, we conclude that

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq c_0.$$

A similar argument shows

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \geq c_0,$$

and, hence, we have (6.19) when $\lambda \geq M(c_0)$ and $c_1 = c_0$.

Now we consider the case where $\lambda \geq M(c_0)$ and $c_1 > c_0$. The definition of c_1 implies that $G^-(c) > c$ for all $c \in [c_0, c_1)$, and, moreover, by the definition of \bar{c} , we have $\bar{c}(\lambda) \in [c_0, c_1]$, $\lambda > M(c)$ for all $c \in [c_0, \bar{c}(\lambda))$, and, if $\bar{c}(\lambda) < c_1$, then $M(\bar{c}(\lambda)) = \lambda$.

Suppose that

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > \bar{c}(\lambda).$$

We assume first that $\bar{c}(\lambda) = c_1$ and observe that we must have $c_1 < g_{\max}$. Then (1.15) yields $\beta_2 \in \mathbb{R}$ so that $G^+(\beta_2) < \beta_2$ and $c_1 < \beta_2 < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$. Fixing $\beta_1 \in (c_1, \beta_2)$, we argue, as in the previous case, with c_1 in place of c_0 and find sequences $\varepsilon_k \rightarrow 0+$, $\{\mu_k\}$ and $\{\lambda_k\}$, and constants $\lambda_0 > 0$ and $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\lambda_0 \leq \mu_k < \lambda_k \leq \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1, \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2,$$

which contradicts Proposition 3.

Assume next that $\bar{c}(\lambda) < c_1$. As noted above, we have $M(\bar{c}(\lambda)) = \lambda$ and $M(c) < \lambda$ for all $c \in [c_0, \bar{c}(\lambda))$, and, in particular,

$$(6.23) \quad M(c) \leq \lambda \quad \text{for all } c \in [c_0, \bar{c}(\lambda)].$$

Since the function \bar{c} is continuous at λ , we may choose $\eta > 0$ so that $\bar{c}(r) < c_1$ for all $r \in [\lambda, \lambda + \eta]$ and noting that, for any $r \in (\lambda, \lambda + \eta]$, $r > M(c_0)$, we find by the definition of $\bar{c}(r)$ that $M(\bar{c}(r)) = r$, which together with (6.23) implies that $\bar{c}(r) > \bar{c}(\lambda)$.

We choose $\gamma \in (0, \eta)$ small enough so that $\bar{c}(\lambda + \gamma) < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$. If we set $\beta_2 = \bar{c}(\lambda + \gamma)$ and fix $\beta_1 \in (\bar{c}(\lambda), \beta_2)$, then we have $\bar{c}(\lambda) < \beta_1 < \beta_2 < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$.

As before, we choose sequences $\{\varepsilon_k\}$, $\{\mu_k\}$ and $\{\lambda_k\}$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and, for some $\lambda_0 > 0$ and for all $k \in \mathbb{N}$,

$$\lambda_0 \leq \mu_k < \lambda_k \leq \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \text{ and } u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

Then Proposition 2 imply that $M(\beta_2) \leq \limsup_{k \rightarrow \infty} \lambda_k \leq \lambda$. On the other hand, we have $M(\beta_2) = M(\bar{c}(\lambda + \gamma)) = \lambda + \gamma > \lambda$. Hence we obtain a contradiction,

Thus, in the case when $\lambda \geq M(c_0)$ and $c_1 > c_0$, we have

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq \bar{c}(\lambda),$$

while, by Lemma 6, we find

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \geq \bar{c}(\lambda),$$

and we conclude that (6.19) holds when $\lambda \geq M(c_0)$ and $c_1 > c_0$.

A similar argument proves that (6.19) holds when $\lambda \geq M(c_0)$ and $c_1 < c_0$, and the proof is complete. \square

APPENDIX A. A SUBSOLUTION PROPERTY

For $T > 0$ and a (relatively) open subset Π of $\partial\Omega$, we consider the problem

$$(A.1) \quad \begin{cases} U_t \leq b(x) \cdot DU & \text{in } \Omega \times (0, T], \\ \min\{U_t - b(x) \cdot DU, U\} \leq 0 & \text{on } \Pi \times (0, T]. \end{cases}$$

Lemma A.1. *Let $U \in \text{USC}(\bar{Q}_T)$ be a subsolution of (A.1), fix $z \in \Omega^\Pi$ and set*

$$u(t) = U(X(T - t, z), t) \quad \text{for } t \in [0, T].$$

Then $u \in \text{USC}([0, T])$ and, if $z \in \Omega$, it is a subsolution of

$$(A.2) \quad u' \leq 0 \quad \text{in } (0, T]$$

and, if $z \in \Pi$, it is a subsolution of

$$(A.3) \quad \begin{cases} u' \leq 0 & \text{in } (0, T), \\ \min\{u', u\} \leq 0 & \text{on } \{T\}. \end{cases}$$

We note that observations like the lemma above concerning the restriction of viscosity solutions to lower dimensional manifolds go back to Crandall and Lions [4, Proposition I.13].

Proof. Let $\phi \in C^1((0, T])$ and assume that $u - \phi$ has a strict maximum at $\hat{t} \in (0, T]$.

For $\alpha > 0$ consider the function $\Phi : \bar{Q}_T \rightarrow \mathbb{R}$ given by

$$\Phi(x, t) := U(x, t) - \phi(t) - \alpha|x - X(T - t, z)|^2,$$

let $(x_\alpha, t_\alpha) \in \bar{Q}_T$ be a maximum point of Φ , set $\hat{x} = X(T - \hat{t}, z)$, and observe that, as $\alpha \rightarrow \infty$, $(x_\alpha, t_\alpha) \rightarrow (\hat{x}, \hat{t})$, $\alpha|x_\alpha - X(T - t_\alpha, z)|^2 \rightarrow 0$ and $U(x_\alpha, t_\alpha) \rightarrow U(\hat{x}, \hat{t})$.

Then, for α sufficiently large, we may assume that $(x_\alpha, t_\alpha) \in \Omega \times (0, T]$ if either $z \in \Omega$ or $\hat{t} < T$, and $(x_\alpha, t_\alpha) \in \Omega^\Pi \times (0, T]$ if $z \in \Pi$.

If $(x_\alpha, t_\alpha) \in \Omega \times (0, T]$, (A.1) yields

$$\phi'(t_\alpha) - 2\alpha(X(T - t_\alpha, z) - x_\alpha) \cdot \dot{X}(T - t_\alpha, z) \leq 2\alpha b(x_\alpha) \cdot (x_\alpha - X(T - t_\alpha, z)),$$

and then

$$\begin{aligned}\phi'(t_\alpha) &\leq 2\alpha(x_\alpha - X(T - t_\alpha, z)) \cdot (b(x_\alpha) - b(X(T - t_\alpha, z))) \\ &\leq 2\|Db\|_{L^\infty(\Omega)}\alpha|x_\alpha - X(T - t_\alpha, z)|^2.\end{aligned}$$

Similarly, if $(x_\alpha, t_\alpha) \in \Pi \times (0, T]$, then we get

$$\phi'(t_\alpha) \leq 2\|Db\|_{L^\infty(\Omega)}\alpha|x_\alpha - X(T - t_\alpha, z)|^2 \quad \text{or} \quad U(x_\alpha, t_\alpha) \leq 0.$$

Sending $\alpha \rightarrow \infty$ yields

$$\phi'(\hat{t}) \leq 0 \quad \text{if either } z \in \Omega \text{ or } \hat{t} < T,$$

and

$$\phi'(\hat{t}) \leq 0 \quad \text{or} \quad u(\hat{t}) \leq 0 \quad \text{if } z \in \Pi \text{ and } \hat{t} = T.$$

□

APPENDIX B. THE SUPERSOLUTION PROPERTY UP TO THE BOUNDARY

For $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$ and $H(x, p) = \alpha(x)p \cdot p + b(x) \cdot p$ we consider the equation

$$(B.1) \quad H(x, Du) = 0 \quad \text{in } \Omega.$$

Lemma B.1. *The maximal subsolution $V \in \text{Lip}(\overline{\Omega})$ of (B.1) with $V(0) = 0$ satisfies, in the viscosity sense,*

$$H(x, DV) \geq 0 \quad \text{on } \overline{\Omega}.$$

Note that the importance of the lemma above is that the viscosity inequality holds up to the boundary.

Proof. Let $\phi \in C^1(\overline{\Omega})$ and assume that $V - \phi$ has a strict minimum at $\hat{x} \in \overline{\Omega}$ and $V(\hat{x}) = \phi(\hat{x})$.

To prove the assertion of the lemma, we argue by contradiction and suppose that $H(\hat{x}, D\phi(\hat{x})) < 0$.

Indeed, if $\hat{x} = 0$, then

$$H(\hat{x}, D\phi(\hat{x})) = \alpha(0)D\phi(0) \cdot D\phi(0) \geq 0,$$

and, henceforth, we may assume that $\hat{x} \neq 0$.

We may choose constants $r > 0$ and $\varepsilon > 0$ so that $0 \notin B_r(\hat{x})$ and

$$(B.2) \quad H(x, D\phi(x)) \leq 0 \quad \text{for all } x \in \overline{\Omega} \cap B_r(\hat{x}),$$

$$(B.3) \quad \varepsilon + \phi(x) < V(x) \quad \text{for all } x \in \overline{\Omega} \setminus B_r(\hat{x}).$$

It follows from (B.2) that, in the viscosity sense,

$$H(x, D\phi) \leq 0 \quad \text{in } \Omega \cap B_r(\hat{x}).$$

Set

$$W(x) = \max\{V(x), \varepsilon + \phi(x)\} \quad \text{for } x \in \overline{\Omega},$$

and observe that $\Omega = N \cup M$, where $N := \Omega \cap B_r(\hat{x})$ and $M := \{x \in \Omega : V(x) > \varepsilon + \phi(x)\}$ (note that N, M are both open subsets of Ω),

$$H(x, DW) \leq 0 \quad \text{in } N \quad \text{in the viscosity sense,}$$

$W = V$ in M and $\hat{x} \in M$. Hence, W is a subsolution of (B.1) such that $W(0) = V(0) = 0$ and $W(\hat{x}) > V(\hat{x})$, which contradicts the maximality of V . □

APPENDIX C. A COMPARISON THEOREM

We follow the arguments of [10, Corollary 2.2 & Remark 2.4] to give a proof of following lemma.

Lemma C.1. *Let $\alpha \in C(\mathbb{R}^n, \mathbb{S}^n(\theta_0))$ and $H(x, p) = \alpha(x)p \cdot p + b(x) \cdot p$. If $v \in \text{Lip}(\overline{\Omega})$ and $w \in \text{LSC}(\overline{\Omega})$ are respectively a subsolution and a supersolution of the state-constraints problem*

$$H(x, Du) = 0 \quad \text{in } \Omega,$$

that is, v and w satisfy, respectively,

$$H(x, Dv) \leq 0 \quad \text{in } \Omega \quad \text{and} \quad H(x, Dw) \geq 0 \quad \text{on } \overline{\Omega},$$

and $v(0) \leq w(0)$, then $u \leq v$ on $\overline{\Omega}$.

Note that the viscosity property of v and w at the origin is indeed not required in the lemma above. That is, it is enough to assume that v and w are a subsolution of

$$H(x, Dv) \leq 0 \quad \text{in } \Omega \setminus \{0\},$$

and a supersolution of

$$H(x, Dw) \geq 0 \quad \text{on } \overline{\Omega} \setminus \{0\}.$$

Proof. Fix $\varepsilon > 0$ and choose $r \in (0, r_0)$ sufficiently small so that

$$\max_{\partial B_r} v \leq \min_{\partial B_r} w + \varepsilon,$$

set $\Omega(r) := \Omega \setminus \overline{B}_r$, define $h \in C(\partial\Omega(r))$ and $v_\varepsilon \in \text{Lip}(\overline{\Omega})$ by

$$v_\varepsilon = v - \varepsilon \quad \text{and} \quad h(x) = \begin{cases} \min_{\partial B_r} w & \text{if } x \in \partial B_r, \\ \max_{\partial\Omega} v & \text{if } x \in \partial\Omega, \end{cases}$$

and observe that v_ε and w are, respectively, a subsolution and a supersolution of the Dirichlet problem in the viscosity sense (see [10]):

$$\begin{cases} H(x, Du) = 0 & \text{in } \Omega(r), \\ u = h \quad \text{or} \quad H(x, Du) = 0 & \text{on } \partial\Omega(r). \end{cases}$$

It follows from [11, Corollary 4] that there exists $\psi \in \text{Lip}(\overline{\Omega}(r))$ which is a subsolution of $H(x, D\psi) \leq -\eta$ in $\Omega(r)$ for some $\eta > 0$ and note that we may assume by adding, if necessary, a constant that $\psi \leq v_\varepsilon$ on $\Omega(r)$.

Define $v^\varepsilon \in \text{Lip}(\overline{\Omega}(r))$ by $v^\varepsilon(x) = (1 - \varepsilon)v_\varepsilon(x) + \varepsilon\psi(x)$ and note that v^ε is a subsolution of

$$\begin{cases} H(x, Du) \leq -\varepsilon\eta & \text{in } \Omega(r), \\ u \leq h \quad \text{or} \quad H(x, Du) \leq -\varepsilon\eta & \text{on } \partial\Omega(r). \end{cases}$$

It is clear that the domain $\Omega(r)$ satisfies the uniform interior cone condition and, hence, we apply [10, Corollary 2.2 & Remark 2.4] to v^ε and w_ε , to conclude that $v^\varepsilon \leq w_\varepsilon$ in $\overline{\Omega}(r)$, from which, after sending $\varepsilon \rightarrow 0$, we get $v \leq w$ on $\overline{\Omega}$. \square

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